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ABSTRACT

This Comprehensive School Mathematics Program (CSMP) guide is divided into three major parts. The first, the Languages of Strings and Arrows, opens with a suggested lesson order. Major sections cover instructional approaches for: (1) Games with Strings, (2) Necklaces, and (3) The Table Game. A Series of appendices is included. Part 2, Geometry and Measurement, covers a suggested lesson order then looks at Geometric Constructions and a Grid Game. Probability and Statistics is the final major section. It also opens with a suggested lesson order then covers lessons grouped under, (1) Random Number Stories, (2) Probability Stories, and (3) Probability Games. (MP)

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Comprehensive School Mathematics Program
CEMREL, Inc.

Experimental Version

8-37981

CSMP Mathematics for the Intermediate Grades
Part V

Teacher's Guide

The Languages of Strings and Arrows
Geometry and Measurement
Probability and Statistics

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THE LANGUAGES OF STRINGS AND ARROWS

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THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings is the nonverbal language of sets, that is, collections of objects put together in some way. The language of arrows is the nonverbal language of relations among objects. These languages permeate the CSMP content strands and are used, separately or together, to present an unending variety of numerical and nonnumerical situations.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, into sets. If the mathematics of sets has any place in an elementary school mathematics curriculum, it is precisely because it can help students to understand and use the ideas of classification.

The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting -- "in" or "out" -- neatly by using string pictures to illustrate "in" and "out" in a dramatic way. Objects to be sorted are represented by dots and the sets into which they are sorted are represented by drawing "strings" around dots. A dot inside (the region delineated by) a set's string is for an object in the set and a dot outside a set's string is for an object not in the set. (See for example the storybook WHERE'S MY NOSE?.)

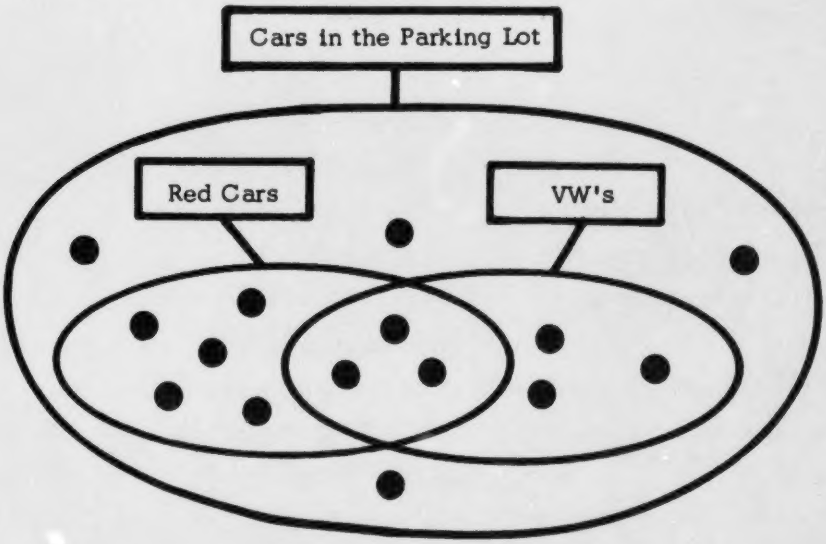
This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The abilities to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity, in general, and for the pursuit and understanding of mathematics, in particular. Ordinarily, we are not expected to deal with such problems until our command of our natural language is fully developed and until we face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires a special mode of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes and to report their thinking long before they have extensive verbal skills.

In this strand we present situations and ask carefully phrased questions to continue to advance skills in classification, always remembering that the skills grow out of such experiences. To be able to draw strings and dots is not an objective in itself; to develop the mode of thinking involved is the objective. Thus it is important for us to construct the situations carefully. The sets into which we ask students to classify objects must be determined by well-defined attributes; otherwise, the students have the added problem of deciding whether a certain object does or does not have a certain attribute.

For example, consider this situation. A teacher announces that the blue string on the board is for all children in the class with dark hair and the red string is for all children in the class who are tall. The teacher invites John, a student in the class, to locate a dot for himself in the picture. He places a dot inside the blue string and outside the red string and labels it "J". There is a flurry of mixed agreement and disagreement in the classroom. John has brown hair and he thinks it is dark, but some say it is really light. John does not think he is tall, but someone insists that he is taller than most of the other children in the class. As a result, most of the energy in the class is directed toward making a decision about what attributes John has in this situation rather than how to classify John relative to those attributes. More care in choice of attributes would have avoided such distraction and confusion. For example, if the blue string were for all children in the class whose birthdays are in January and the red string for all the children who have a sister, John's choice of a dot for himself would be criticized strictly on its placement (i.e., his classification of himself) and not on his decision as to what attributes he has.

For this reason we make extensive use of A-Blocks (Attribute Blocks) and sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, and so on. They know whether or not a given number is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to count the objects that have a certain attribute. Suppose for example, in a parking lot there are eight red cars and six Volkswagens. If four cars in the lot are neither red nor Volkswagens, can there be eighteen cars in the lot? Suppose we count the cars in the parking lot and find there are exactly fifteen. Is this possible? A string picture immediately settles this apparent paradox.



There are three red Volkswagens.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in nonmathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphical representation of a relation, somewhat the same way that a blueprint is a handy graphical representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way -- usually more conveniently than the same information could be presented in words. (See for example the lessons on the storybook A VALENTINE MYSTERY.)

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read" -- and also draw -- an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical, motor problems of writing or reading about the ideas in conventional systems. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Another educational bonus occurs when arrow diagrams are used imaginatively. A story or problem may be captured by an arrow diagram, and at the same time, the resulting arrow diagram may suggest other, similar situations. This allows students to call upon previous experience or to expand their creativity.

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. We resist making the ability to draw prescribed arrows an objective in itself, just as we resist viewing an arrow diagram as just another format for drill problems in arithmetic.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought-provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze.

By this time your students should be very familiar and quite comfortable with both the language of strings and the language of arrows. The content of this strand for the present semester builds in a variety of ways on this fluency with the languages. Each of the three chapters included in this strand presents situations quite distinct from the others.

Chapter 1 introduces another kind of string game — the Divisors Game. Each lesson looks at this new game in some way and then continues with a systematic analysis of the String Game.

Chapter 2 presents a special problem that touches upon many areas of the program but, in particular, develops a language — a code — to assist in the solution.

Chapter 3 prepares for and introduces the Table Game to develop strategic thinking in another setting. But there are many side benefits. The game provides practice with tables for operations, reviews some operations previously introduced in the World of Numbers strand, and introduces some new operations which are interesting mathematically. This game will be explored further in Part VI.

1

GAMES WITH STRINGS

The Language of Strings is used as a vehicle to present various situations involving sets which stimulate logical thinking. Part V continues the experiences with the String Game, thoroughly familiar to students from preceding parts of CSMP Mathematics for the Intermediate Grades, and introduces the Divisors Game, a definite relative of the former. The object in this new game is still to identify the strings, but this time knowing that each string is for the positive divisors of a non-zero whole number less than or equal to 50.

The String Game situations involve students in thinking critically about the relations "is a divisor of", "is a multiple of", "is larger than", etc., as well as challenge them to think logically about the relationships between various sets and to give persuasive arguments. In the Divisors Game, students specifically investigate the relation "is a divisor of". Strategies involve thinking about which numbers are most beneficial to locate in the string picture, about where multiples of a number already located could or do belong in the picture, and about the implications of knowing the secret numbers must be 50 or smaller.

The situations are rich; they encourage deductive reasoning and further develop the students' familiarity and ease with numerical relations.

CAPSULE LESSON SUMMARIES

Section 1.1: Put numbers, named in a variety of ways, such as:

75, 3×17, $^{-}5$, $16 + 88$,

in a two string picture where each string is for the multiples of some known number. Play the String Game with two strings in the usual way.

Section 1.2: Introduce the Divisors Game, a type of string game in which each of two strings is for the positive divisors of an unknown number. Through the selection and the placement of numbers in the string picture, discover the unknown numbers. Either do a variation of the game involving a three-string picture or put numbers in a two-string picture revealing the string labels as the positive divisors of 36 and the positive divisors of 42.

Section 1.3: Play the Divisors Game with two strings and possibly with three strings. In the context of the String Game, use clues involving the placement of certain numbers (where each belongs or does not belong) to determine the labels of two strings. Play the String Game in the usual way or with special scoring rules.

Section 1.4: Play the Divisors Game with two strings. Whenever a number is located in the picture, discuss the information gained about the string labels. In the context of the String Game, use clues involving the placement of certain numbers and a clue telling how many numbers belong in a particular region to determine the labels of two strings. Either do a similar activity with clues given about a three-string picture or place the game pieces in the picture with the string cards face-up.

Section 1.5: In the context of the Divisors Game, use clues involving the placement of certain numbers to determine the labels of two strings. Do a similar activity in the context of the String Game. Play the String Game with three strings in the usual way or place the game pieces in the picture with the string cards face-up.

Section 1.6: Play the Divisors Game. In the context of the String Game, use clues involving the placement of certain numbers to determine the labels of two strings. (Worksheets L01, L02)

1.1 Games with Strings #1

CAPSULE LESSON SUMMARY

Put numbers, named in a variety of ways, such as:

75, 3 x 17, $^{-}5$, 16 + 88,

in a two-string picture where each string is for the multiples of some known number. Play the String Game with two strings in the usual way.

MATERIALS

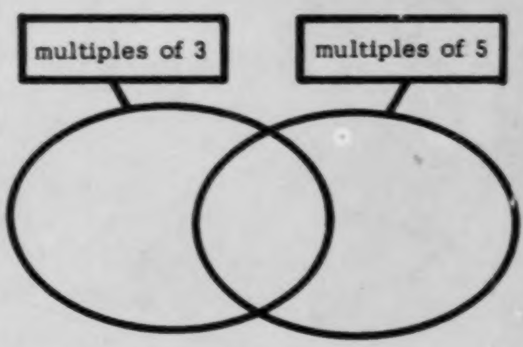
Teacher: Colored chalk; Numerical String Game kit
Student: None

DESCRIPTION OF LESSON

Exercise 1

Draw two large overlapping strings; label one string "multiples of 3" and the other string "multiples of 5". To one side of the string picture, draw a Team Board with the indicated numbers on the respective sides of the board.

TEAM A		TEAM B	
3×17	29	3×4	$^{-}3$
75	2×33	60	4×60
3×95	90	2×75	13
100	5×13	50	5×19
25	3	35	$^{-}5$
$^{-}12$	67	45	22



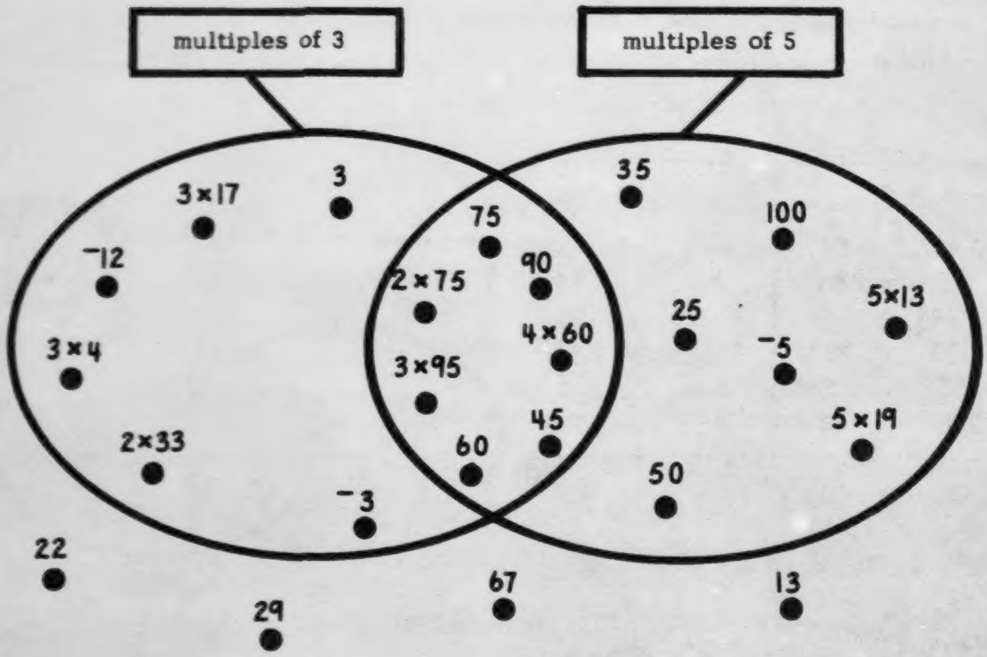
L 1

Divide your class into two teams, Team A and Team B.

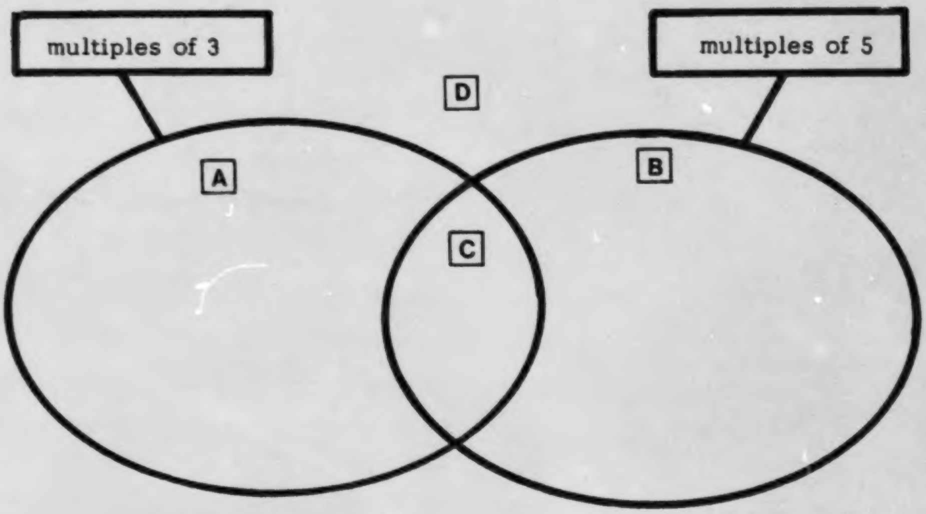
T: We are going to play a string game with these numbers. During your turn, you will have only one chance to put a number in the string picture. The first team to put all the numbers from its side of the board correctly in the picture wins.

Alternating teams, call on students to locate numbers in the picture and to write them in chalk. If a number is correctly placed in the picture, erase it from the Team Board; if a number is incorrectly placed, erase it from the string picture.

The next illustration shows the correct placement of all numbers from both sides of the Team Board.


multiples of 3
multiples of 5
3×17
3
⁻12
3×4
2×33
⁻3
75
2×75
90
4×60
3×95
45
60
35
100
25
5×13
⁻5
5×19
50
22
29
67
13

When one team has won the game, consider any numbers not yet placed in the picture from the other team's side of the board. Let the students discuss how they know where various numbers belong in the picture. Ask the following questions about the picture. (The next illustration shows each of the four regions with a letter label. These labels only make the lesson description easier to read; do not write them on the board.)

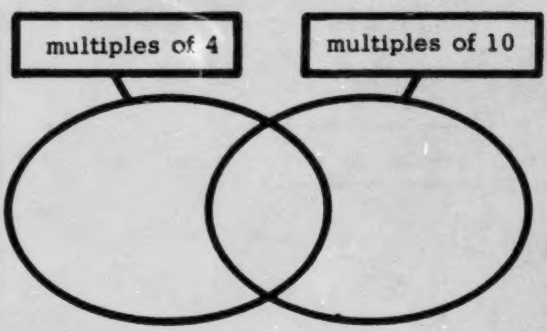


- Where does three times any integer belong in the picture? (In the red string)
- Which multiples of 3 are in Region C? Which are in Region A? (Three times any multiple of 5 is in Region C; three times any integer which is not a multiple of 5 is in Region A)
- Where does five times any integer belong in the picture? (In the blue string)
- Which multiples of 5 are in Region C? Which are in Region B? (Five times any multiple of 3 is in Region C; five times any integer which is not a multiple of 3 is in Region B)
- Which numbers are in Region D? (Numbers which are neither multiples of 3 nor multiples of 5)

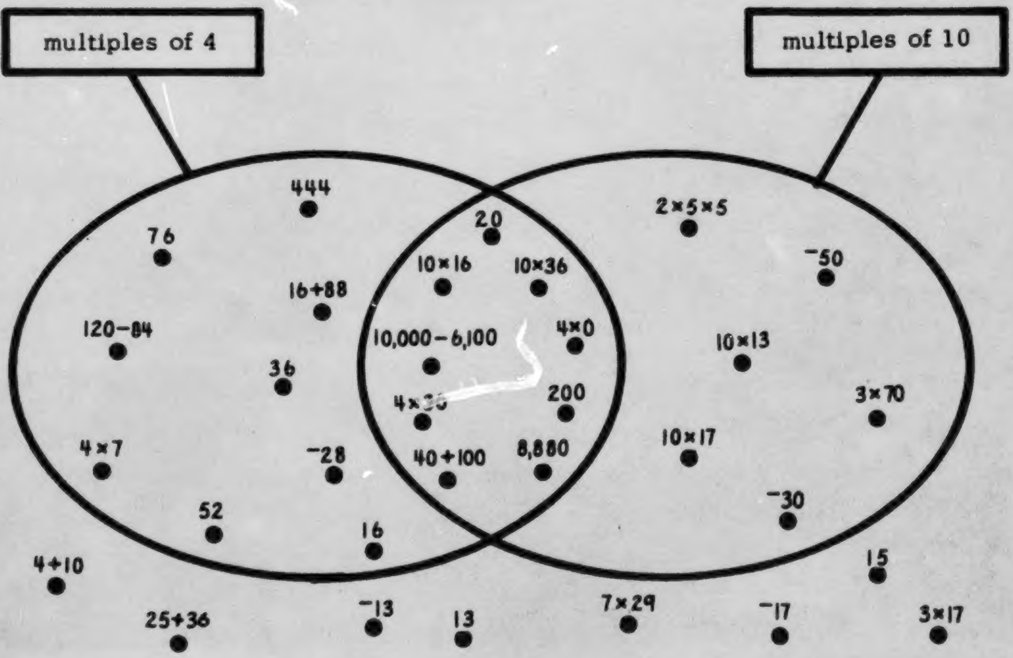
L1

Repeat this activity for the situation pictured below.

TEAM A		TEAM B	
20	15	16	$^{-}28$
4+10	4×30	200	3×70
$^{-}13$	36	10×36	52
10×16	7×29	13	3×17
4×7	$^{-}50$	25+36	$^{-}30$
76	10×17	4×0	40+100
2×5×5	16+88	$^{-}17$	10×13
10,000−6,100	444	8,880	120−84


multiples of 4
multiples of 10

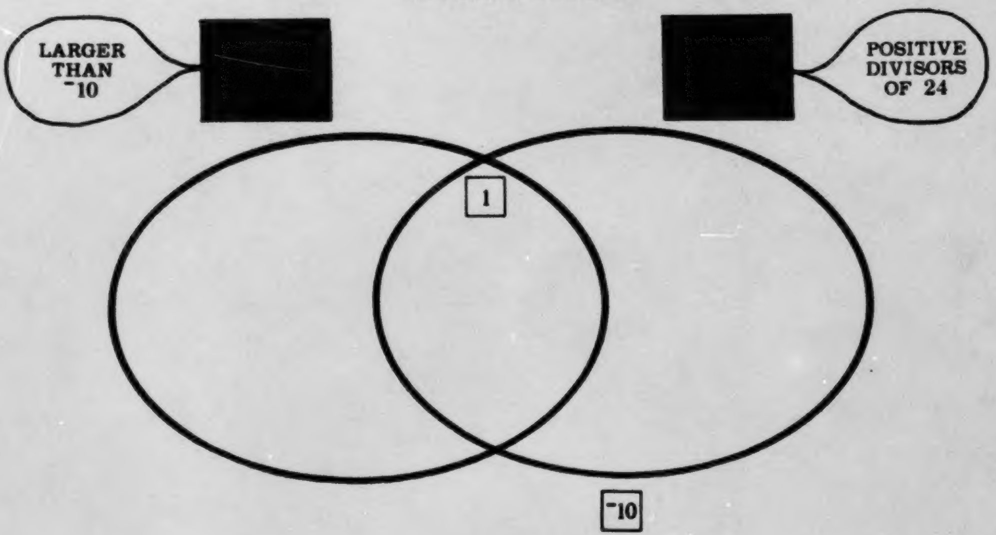
The next illustration shows the correct placement of all the numbers in the string picture.


multiples of 4
multiples of 10
444
76
16+88
120−84
36
4×7
$^{-}28$
52
16
20
10×16
10×36
10,000−6,100
4×0
4×30
200
40+100
8,880
2×5×5
$^{-}50$
10×13
3×70
10×17
$^{-}30$
15
4+10
25+36
$^{-}13$
13
7×29
$^{-}17$
3×17

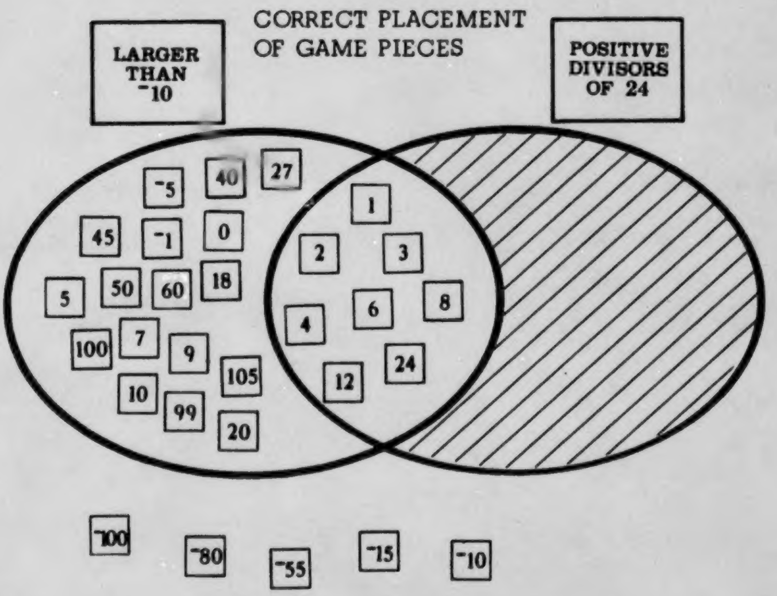
Exercise 2

Play the String Game in the usual way (see the Appendix: The String Game, Version D, Variation #1) in any remaining time. A possible starting situation is illustrated below.

STARTING CLUES


LARGER THAN ⁻10
POSITIVE DIVISORS OF 24
1
⁻10

CORRECT PLACEMENT OF GAME PIECES


LARGER THAN ⁻10
POSITIVE DIVISORS OF 24
⁻5
40
27
45
⁻1
0
5
50
60
18
100
7
9
105
10
99
20
1
2
3
4
6
8
12
24
⁻100
⁻80
⁻55
⁻15
⁻10

L 2

1.2 Games with Strings #2

CAPSULE LESSON SUMMARY

Introduce the Divisors Game, a type of string game in which each of two strings is for the positive divisors of an unknown number. Through the selection and the placement of numbers in the string picture discover the unknown numbers. Either do a variation of the game involving a three string picture or put numbers in a two string picture revealing the string labels as the positive divisors of 36 and the positive divisors of 42.

MATERIALS

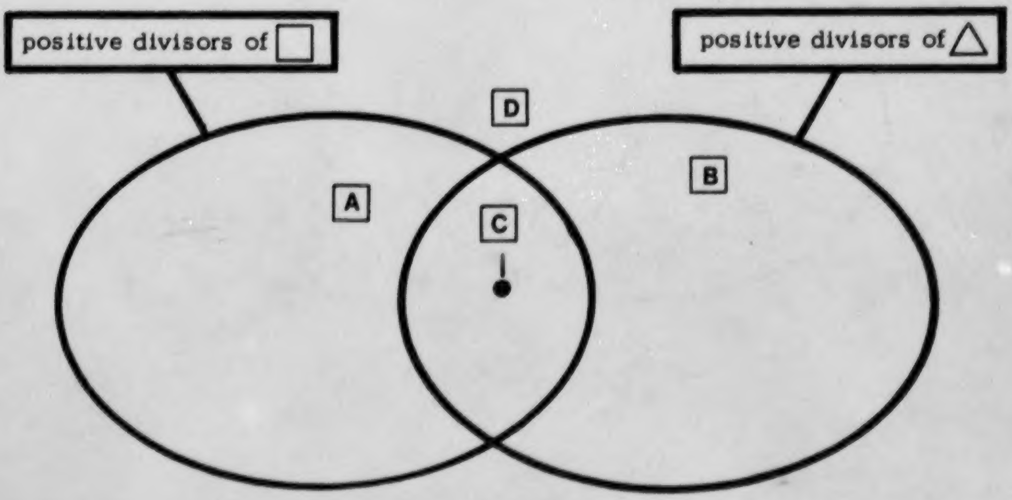
Teacher: Colored chalk
Student: None

DESCRIPTION OF LESSON

If most of your students are average or above, do Exercises 1 and 2A. Otherwise, do Exercises 1 and 2B.

Exercise 1: The Divisors Game

Draw this string picture on the board. The four regions of the picture have been labeled "A", "B", "C", and "D" in the illustration to make it easier to refer to them in the lesson description. Do not write these letters on the board.



Play a type of string game in which students eventually determine the string labels. The numbers which belong in the square and in the triangle can be the same or different but are restricted to whole numbers from 1 to 50. The number 1 is initially placed in the middle region because 1 is a positive divisor of any whole number.

Divide the class into teams. The teams alternate, and the members take turns within each team. During a turn, a student either can select a whole number from 2 to 50 for you to locate in the picture or can try to identify one of the strings. If a student correctly identifies a string label, you record the appropriate number in the square or in the triangle. Otherwise, simply indicate that the identification is incorrect.

Points are gained or lost during the game in the following ways.

- A team gains one point if a member selects a number which belongs in Region A or in Region B.
- A team gains two points if a member selects a number which belongs in Region C.
- A team neither gains nor loses points if a member selects a number which belongs in Region D.
- A team gains four points if a member correctly identifies one of the strings.
- A team loses one point if a member incorrectly identifies one of the strings.

The game is over when both strings have been determined. Tally the scores for the teams; the team with the most points wins the game.

The following dialogue is a condensed description of a game played by two teams in one of our development classes. In this game the red string is for positive divisors of 30 and the blue string is for positive divisors of 42. See page 12 for a partial crib sheet for this game.

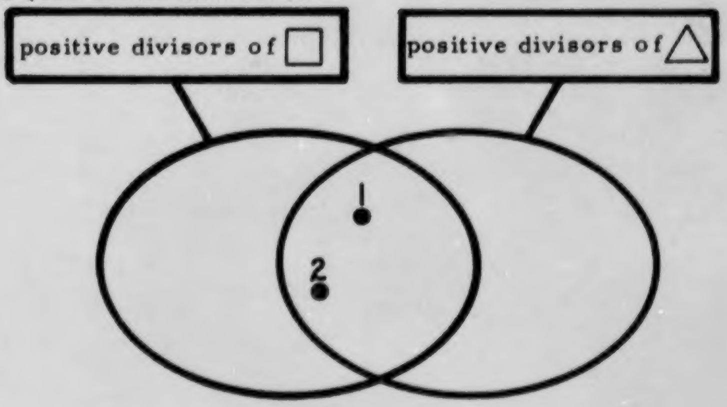
L 2

S: 2.

T: 2 belongs here (see the illustration).

SCORE

TEAM A	TEAM B
2	

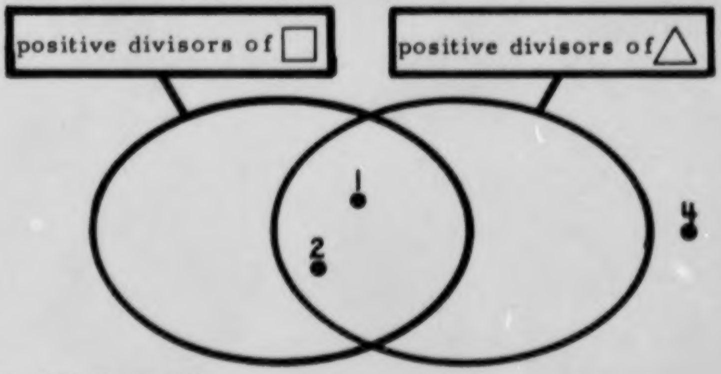

positive divisors of □
positive divisors of △
1
2

S: 4.

T: 4 belongs here (see the illustration).

SCORE

TEAM A	TEAM B
2	


positive divisors of □
positive divisors of △
1
2
4

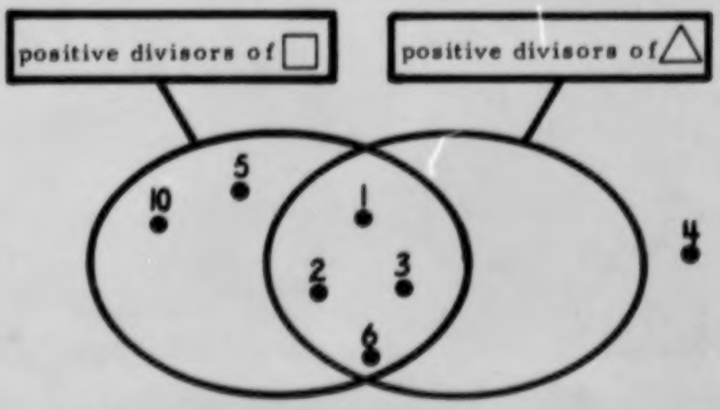
Students next selected 5, 10, 6, and 3.

S. The red string is for positive divisors of 30.

T: That's right !

SCORE

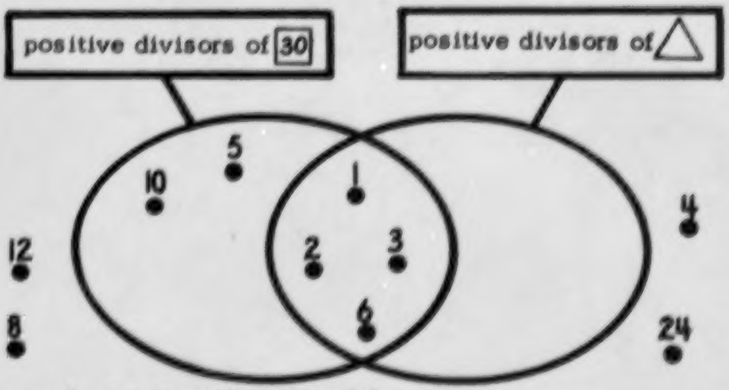
TEAM A	TEAM B
2	1
1	2
2	
4	


positive divisors of □
positive divisors of △
10
5
1
2
3
6
4

Students next selected 24, 8, and 12.

SCORE

TEAM A	TEAM B
2	1
1	2
2	
4	

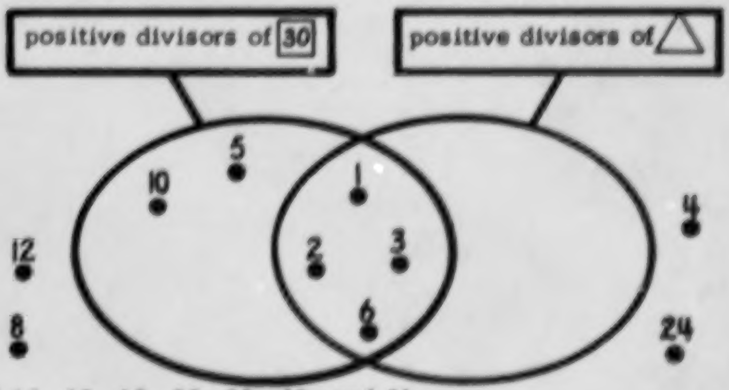

positive divisors of 30
positive divisors of △
5
10
1
2
3
6
12
8
4
24

S: The blue string is for positive divisors of 6.

T: No. Your team loses a point. I'll subtract 1 from your score.

SCORE

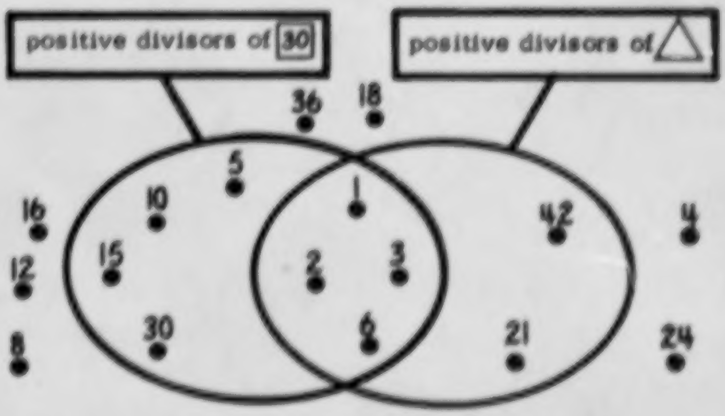
TEAM A	TEAM B
2	1
~~1~~	2
2	
4	


positive divisors of 30
positive divisors of △
5
10
1
2
3
6
12
8
4
24

Students next selected 16, 15, 18, 36, 30, 42, and 21.

SCORE

TEAM A	TEAM B
2	1
~~1~~	2
2	1
4	1
1	
1	


positive divisors of 30
positive divisors of △
36
18
5
10
1
16
15
2
3
42
4
12
30
6
21
24
8

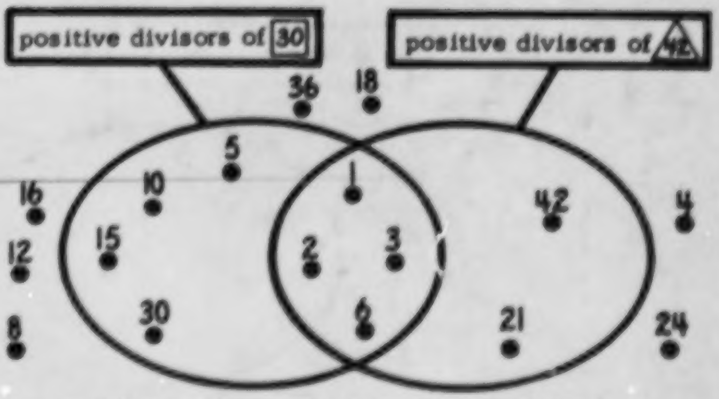
S: The blue string is for positive divisors of 42.

L 2

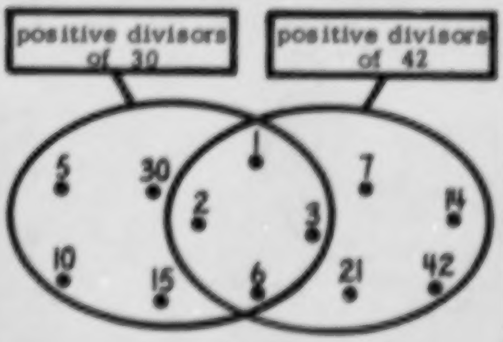
T: That's right!

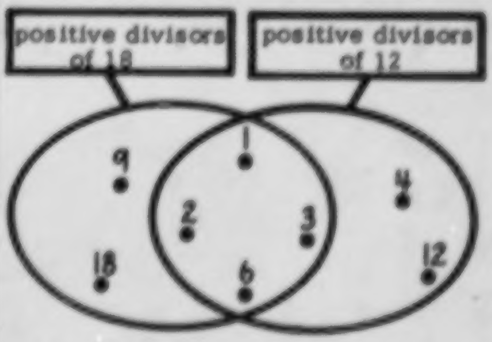
SCORE

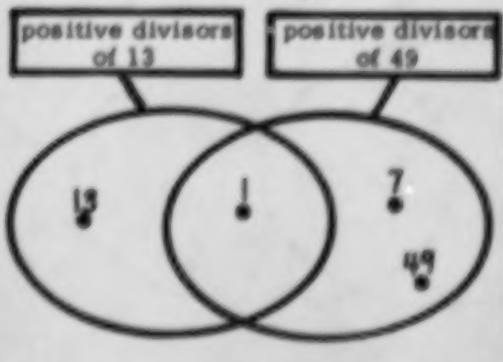
TEAM A	TEAM B
2	1
*	2
2	1
4	1
1	5
1	
4	
14	

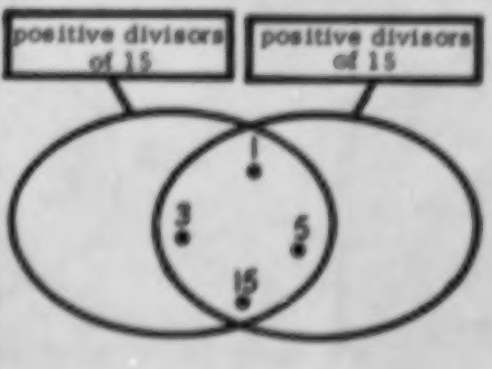

positive divisors of 30
positive divisors of 42
36
18
5
10
16
15
12
8
30
1
2
3
6
42
21
4
24

Play this game several times with your class. Partial crib sheets for four possible games are given below. Note that only the numbers inside the strings are shown; assume that any whole number from 1 to 50 not in the pictures is outside the strings. Remember that the only number put in the picture at the beginning of a game is the number 1.


positive divisors of 30
positive divisors of 42
5
30
10
15
1
2
3
6
7
14
21
42

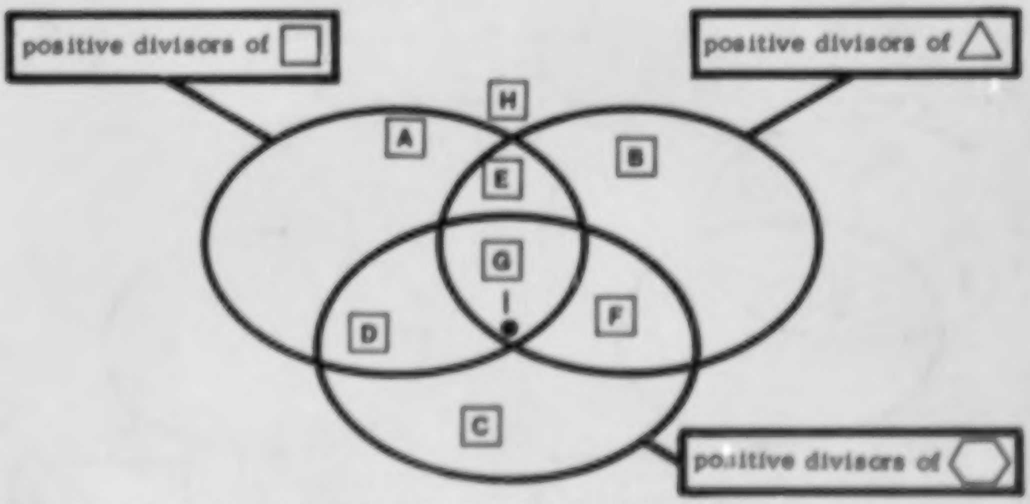

positive divisors of 18
positive divisors of 12
9
18
1
2
3
6
4
12


positive divisors of 13
positive divisors of 49
13
1
7
49


positive divisors of 15
positive divisors of 15
1
3
5
15

Exercise 2A

Play another version of the Divisors Game described in Exercise 1. Introduce a third string into the picture and label it "positive divisors of ⬡ ". Locate the number 1 in the intersection of the three strings before starting the game. The eight regions of the picture have been labeled in the next illustration to facilitate describing how points are gained or lost in the game. Do not write these letters on the board.

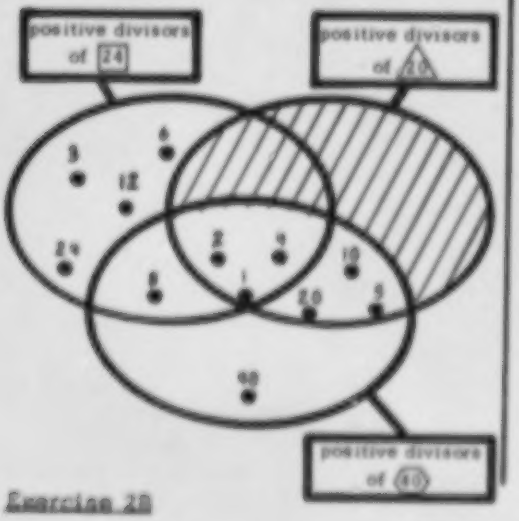


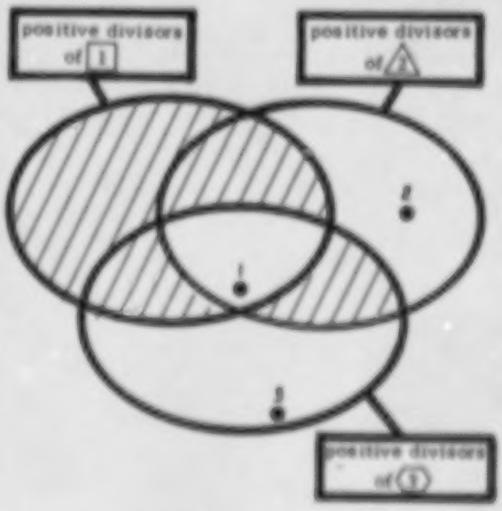
Points are gained or lost during the game in the following ways.

- A team gains one point if a member selects a number which belongs in Region A, B, or C.
- A team gains two points if a member selects a number which belongs in Region D, E, or F.
- A team gains three points if a member selects a number which belongs in Region G.
- A team neither gains nor loses points if a member selects a number which belongs in Region H.
- A team gains five points if a member correctly identifies one of the string labels.
- A team loses one point if a member incorrectly identifies one of the string labels.

The game is over when all three strings have been determined. The team with the most points wins the game.

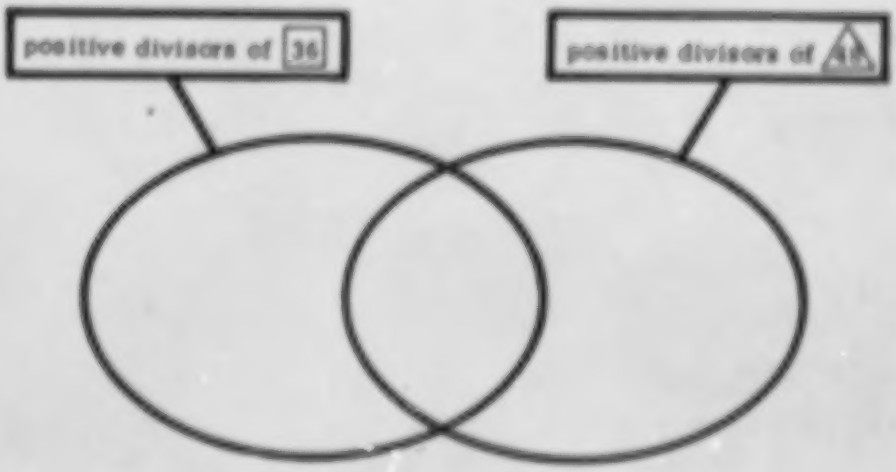
Play this game twice with your class. Partial crib sheets for two possible games are given below. Note that only the numbers inside the strings are shown; assume that any whole number from 1 to 50 not in the pictures is outside the strings. The hatching is included for your reference and is not to be given as a clue.


positive divisors of 24
positive divisors of 20
positive divisors of 40
3
6
12
24
8
2
4
1
10
20
5
40


positive divisors of 1
positive divisors of 2
positive divisors of 3
2
1
3

Exercise 2B

Label the strings in the picture from Exercise 1 in this way.


positive divisors of 36
positive divisors of 48

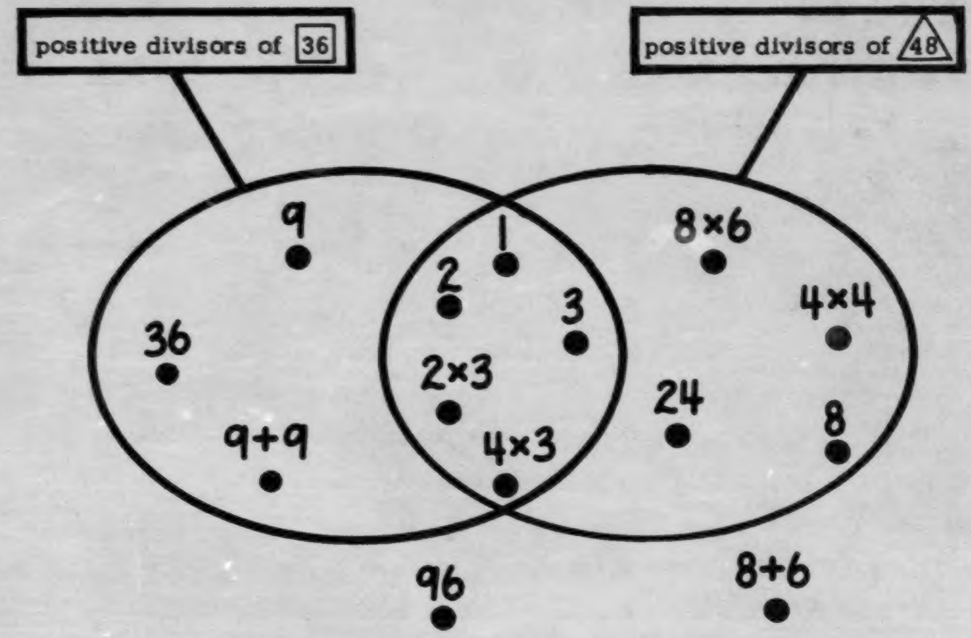
Ask students to explain where each of the following numbers belong in the picture.

1 2 3 9 24

2×3 96 4×4 8×6 9+9

36 4×3 8 8+6

The next illustration shows the correct placement of these numbers.


positive divisors of 36
positive divisors of 48
9
36
9+9
1
2
3
2×3
4×3
8×6
4×4
24
8
96
8+6

1.3 Games with Strings #3

CAPSULE LESSON SUMMARY

Play the Divisors Game with two strings and possibly with three strings. In the context of the String Game, use clues involving the placement of certain numbers (where each belongs or does not belong) to determine the labels of two strings. Play the String Game in the usual way or with special scoring rules.

MATERIALS

Teacher: Colored chalk; Demonstration (3-string) Analysis Sheet; Numerical String Game kit
Student: String Game Analysis Pad

NOTE: The String Game poster for analysis of a 3-string game and the individual 3-string analysis sheets can be used for either a 2-string or a 3-string situation. In the former, simply cross out (or fold under) the third column before starting the activity.

DESCRIPTION OF LESSON

There are three exercises in this lesson and two versions of the second exercise, Exercise 2A and Exercise 2B; choose whichever format is more appropriate for your class.

Exercise 1: The Divisors Game

Divide the class into teams[*] and play the Divisors Game (described in Lesson L1.2) twice. The next illustration provides partial crib sheets for three possible games; choose two of these or make up similar ones. Note that only the numbers inside the strings are shown in the crib sheets; assume that any whole number from 1 to 50 not shown in the pictures is outside the strings. The hatching is included for your reference and is not to be given as a clue. Remember that the only number put in the picture at the beginning of a game is the number 1.

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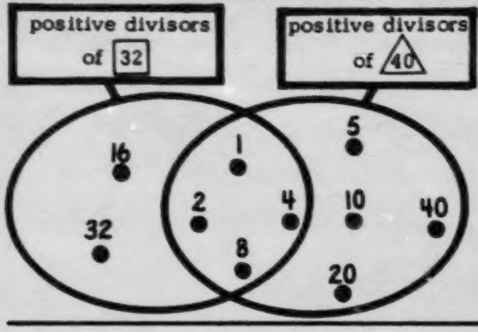
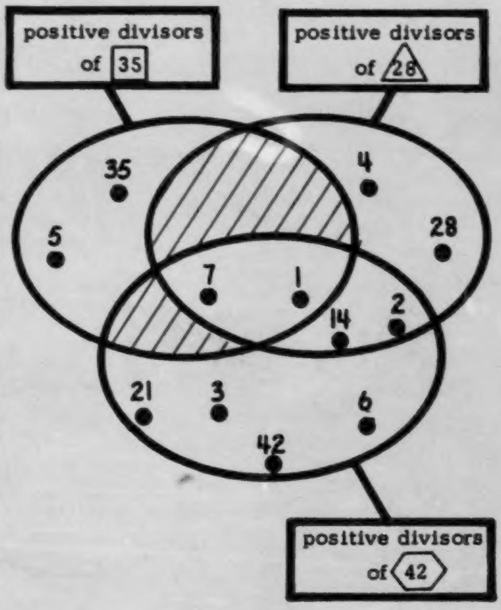
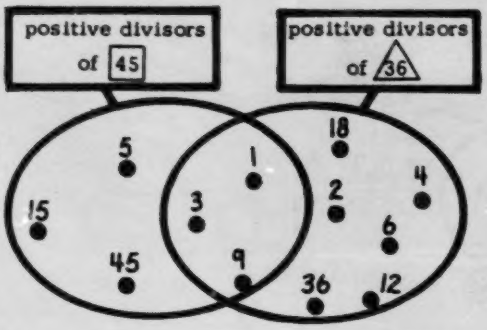
[*] The Divisors Game can be played with two or more teams. With a large class or for variety you may wish to have three teams.


positive divisors of 32
positive divisors of 40
16
32
1
2
4
8
5
10
40
20

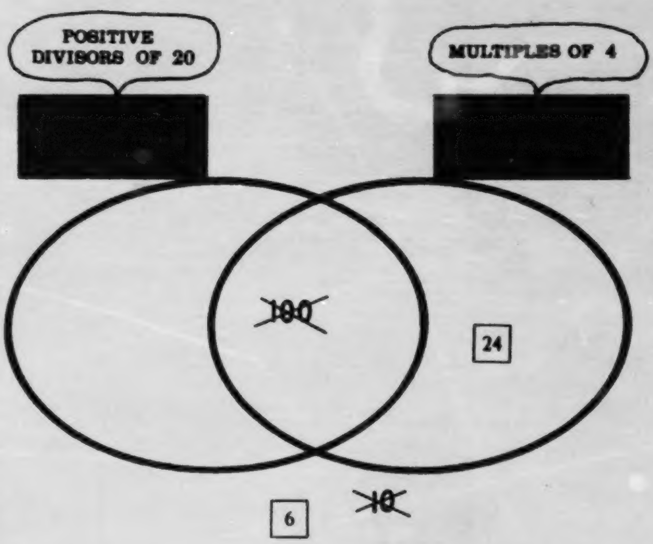

positive divisors of 35
positive divisors of 28
positive divisors of 42
35
5
4
28
7
1
14
2
21
3
6
42


positive divisors of 45
positive divisors of 36
5
15
45
1
3
9
18
2
4
6
36
12

Exercise 2A

Erase the board and prepare for a two-string analysis activity as shown below.

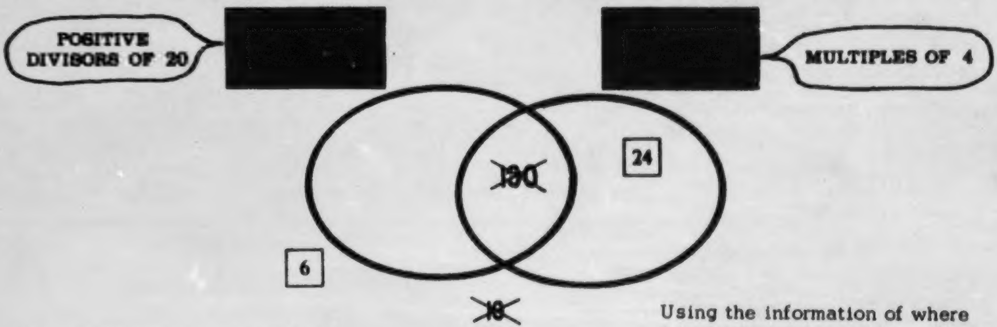
RED	BLUE
MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 50	LARGER THAN 50
SMALLER THAN 50	SMALLER THAN 50
LARGER THAN ⁻10	LARGER THAN ⁻10
SMALLER THAN ⁻10	SMALLER THAN ⁻10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27


POSITIVE DIVISORS OF 20
MULTIPLES OF 4
100
24
6
10

L 5

Distribute sheets from the individual String Game Analysis Pad. Ask the students to cross out labels the strings cannot have and to circle what the strings are for on their lists. If you want to use a game format, give the students a five-to-ten minute time limit in which to discover the string labels. The scoring could be that a team gains five points for each member who identifies only one of the strings correctly and gains fifteen points for each member who identifies both strings correctly.

Collectively analyze the situation.

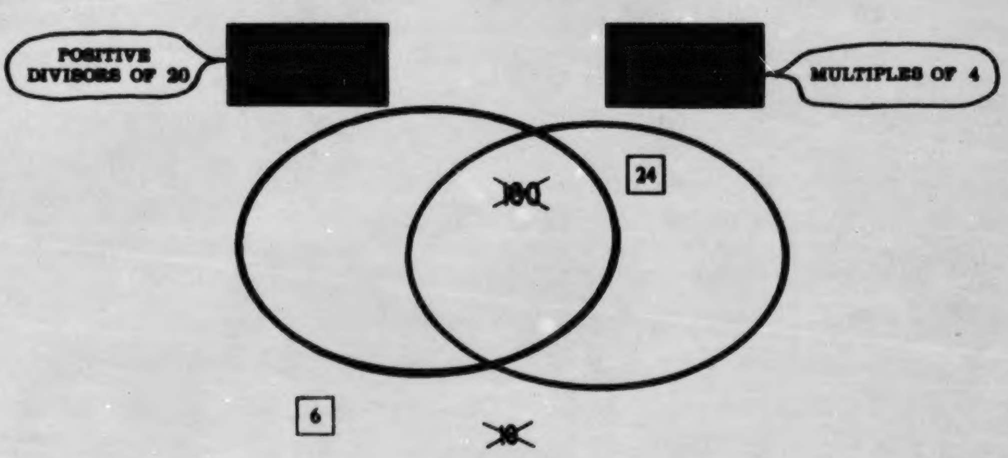


Using the information of where 24 is in the picture, cross off these labels.

RED	BLUE
~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	MULTIPLES OF 3
~~MULTIPLES OF 4~~	MULTIPLES OF 4
MULTIPLES OF 5	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
LARGER THAN 50	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	SMALLER THAN 50
~~LARGER THAN 10~~	LARGER THAN ⁻10
SMALLER THAN ⁻10	~~SMALLER THAN 10~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

Using the information of where 6 is in the picture, cross off additional labels. Find that the blue string is for the multiples of 4.

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	[✓ MULTIPLES OF 4]
MULTIPLES OF 5	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
LARGER THAN 50	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~
SMALLER THAN ⁻10	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~



10 is not a multiple of 4. Using the information that 10 is not outside both strings, decide that 10 belongs inside the red string (but outside the blue string) and cross off additional labels.

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	[✓ MULTIPLES OF 4]
MULTIPLES OF 5	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

100 is a multiple of 4. Using the information that 100 is not in the middle region, decide that 100 is not in the red string. The red string is for the positive divisors of 20.

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	[✓ MULTIPLES OF 4]
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
[✓ POSITIVE DIVISORS OF 20]	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

Exercise 2B

As a group activity, analyze the same situation as in Exercise 2A, except give the four clues one at a time.

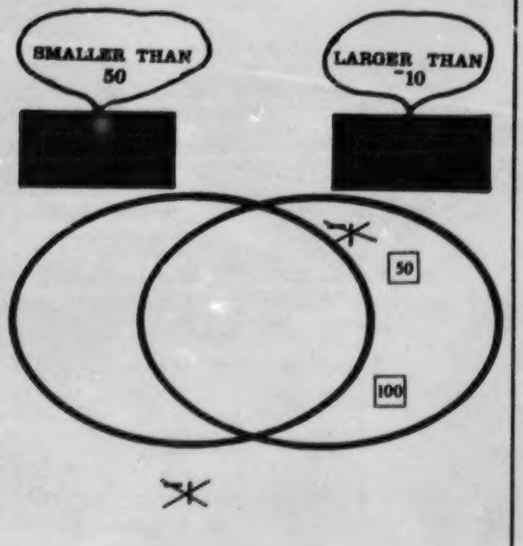
- First, locate the number 24 in the picture.
- Second, locate the number 6 in the picture.
- Third, indicate that the number 10 is not outside both strings.
- Fourth, indicate that the number 100 is not in the middle region.

See the description of Exercise 2A for a reference to which labels can be crossed off the lists after each of the above clues is given.

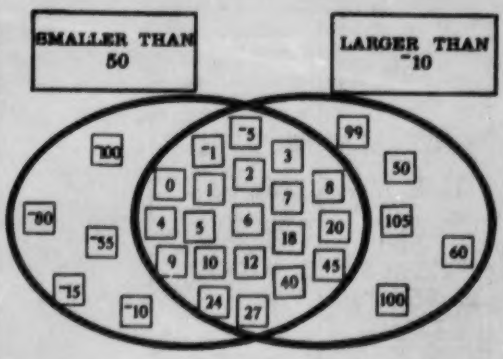
Exercise 3

Play the Numerical String Game with special scoring rules (see the Appendix: The String Game, Version E), or play the game in the usual way. A possible game is suggested below.

STARTING CLUES


SMALLER THAN 50
LARGER THAN ⁻10
50
100

CORRECT PLACEMENT OF GAME PIECES


SMALLER THAN 50
LARGER THAN ⁻10
⁻100
⁻80
⁻35
⁻15
⁻10
⁻5
⁻1
3
0
1
2
8
7
4
5
6
18
20
9
10
12
45
40
24
27
99
50
105
60
100

1.4 Games with Strings #4

CAPSULE LESSON SUMMARY

Play the Divisors Game with two strings. Each time a number is located in the picture, discuss the information gained about the string labels. In the context of the String Game, use clues involving the placement of certain numbers and a clue telling how many numbers belong in a particular region to determine the labels of two strings. Either do a similar activity with clues given about a three string picture or place the game pieces in the picture with the string cards face-up.

MATERIALS

Teacher: Colored chalk; Demonstration (3-string) Analysis Sheets; Numerical String Game kit

Student: String Game Analysis Pad

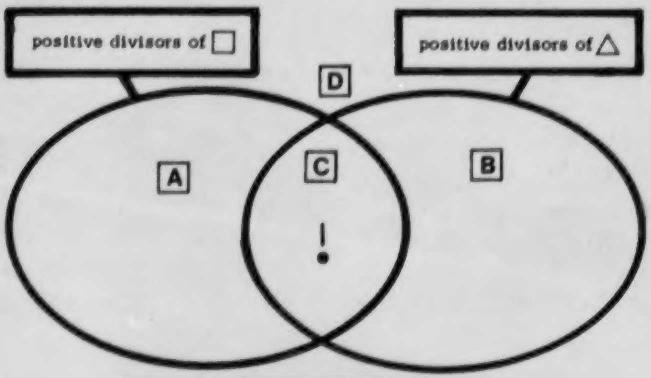
NOTE: The String Game poster for analysis of a 3-string game and the individual 3-string analysis sheet can be used for either a 2-string or a 3-string situation. In the former, simply cross out (or fold under) the third column before starting the activity.

DESCRIPTION OF LESSON

There are three exercises in this lesson, with two versions of the second exercise and of the third exercise. Choose whichever format is most appropriate for your class; Exercise 2B and 3B are simpler versions of Exercises 2A and 3A, respectively.

Exercise 1: The Divisors Game

Prepare your board to play the Divisors Game and list the whole numbers from 1 to 50 to one side of the string picture. Divide the class into two or three teams.



Do not write the letters labeling regions on the board. They are here just to make the description of the lesson easier to read.

1	11	21	31	41
2	12	22	32	42
3	13	23	33	43
4	14	24	34	44
5	15	25	35	45
6	16	26	36	46
7	17	27	37	47
8	18	28	38	48
9	19	29	39	49
10	20	30	40	50

L 7

Play the game in the usual way (see Lesson L1.2) except when a number is located in the picture, collectively discuss which numbers cannot be in "□" and which numbers cannot be in "△". If a number cannot be in "□", slash the numeral in the list using a red marker; if a number cannot be in "△", slash the numeral in the opposite direction using a blue marker. The three types of analysis involved are explained in the examples below.

If 2 is located in Region A, the number in "□" must be a multiple of 2. Odd numbers are not multiples of 2 and so can be slashed in red. The number in "△" cannot be a multiple of 2, so the even numbers can be slashed in blue.

~~1~~	~~11~~	~~21~~	~~31~~	~~41~~
~~2~~	~~12~~	~~22~~	~~32~~	~~42~~
~~3~~	~~13~~	~~23~~	~~33~~	~~43~~
~~4~~	~~14~~	~~24~~	~~34~~	~~44~~
~~5~~	~~15~~	~~25~~	~~35~~	~~45~~
~~6~~	~~16~~	~~26~~	~~36~~	~~46~~
~~7~~	~~17~~	~~27~~	~~37~~	~~47~~
~~8~~	~~18~~	~~28~~	~~38~~	~~48~~
~~9~~	~~19~~	~~29~~	~~39~~	~~49~~
~~10~~	~~20~~	~~30~~	~~40~~	~~50~~

If 3 is located in Region C, then the number in "□" and the number in "△" must both be multiples of 3. Any number which is not a multiple of 3 can be slashed in red and in blue.

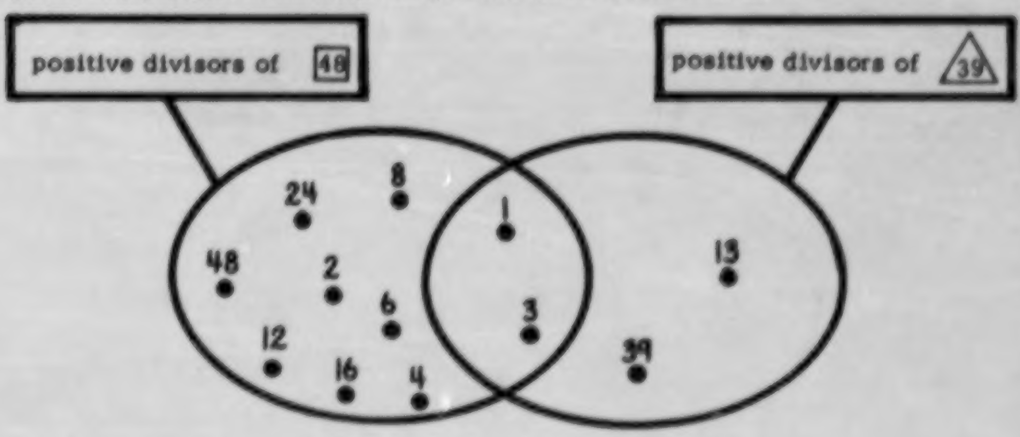
~~1~~	~~11~~	21	~~31~~	~~41~~
~~2~~	12	~~22~~	~~32~~	42
3	~~13~~	~~23~~	33	~~43~~
~~4~~	~~14~~	24	~~34~~	~~44~~
~~5~~	15	~~25~~	~~35~~	45
6	~~16~~	~~26~~	36	~~46~~
~~7~~	~~17~~	27	~~37~~	~~47~~
~~8~~	18	~~28~~	~~38~~	48
9	~~19~~	~~29~~	39	~~49~~
~~10~~	~~20~~	30	~~40~~	~~50~~

If 5 is located in Region D, then the number in "□" and the number in "△" cannot be multiples of 5. Any number which is a multiple of 5 can be slashed in red and in blue.

1	11	21	31	41
2	12	22	32	42
3	13	23	33	43
4	14	24	34	44
~~5~~	~~15~~	~~25~~	~~35~~	~~45~~
6	16	26	36	46
7	17	27	37	47
8	18	28	38	48
9	19	29	39	49
~~10~~	~~20~~	~~30~~	~~40~~	~~50~~

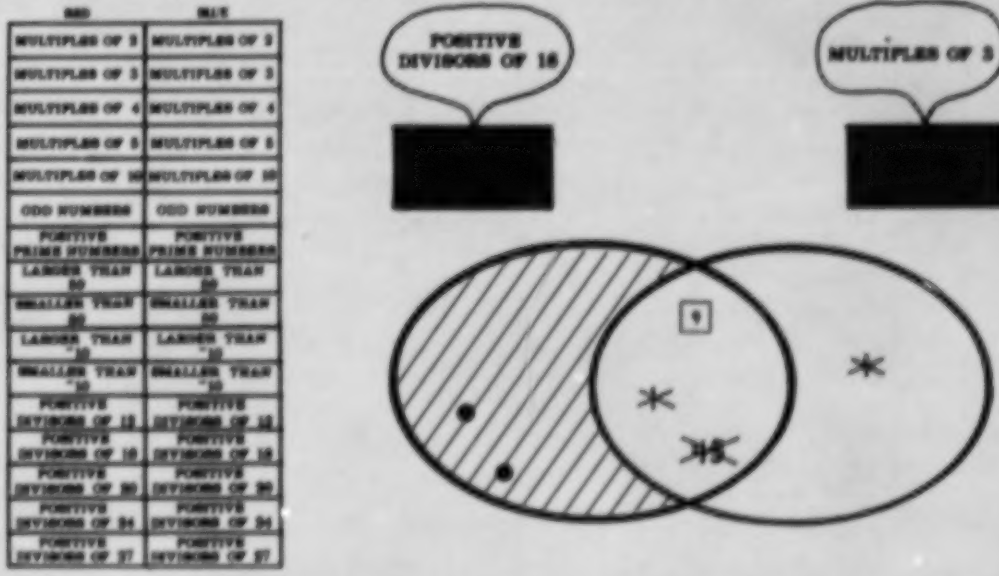
When a student uses a turn to identify the label for one of the strings, ask that student to convince the class using the numbers in the picture. Continue until both strings are determined and correctly identified.

A partial crib sheet is included for a possible game. Note that only the numbers inside the strings are shown; assume that any whole number from 1 to 50 not shown in the picture is outside the strings. Remember that the only number put in the picture at the beginning of the game is the number 1.


positive divisors of 48
positive divisors of 39
24
8
1
48
2
6
3
12
16
4
13
39

Exercise 2A

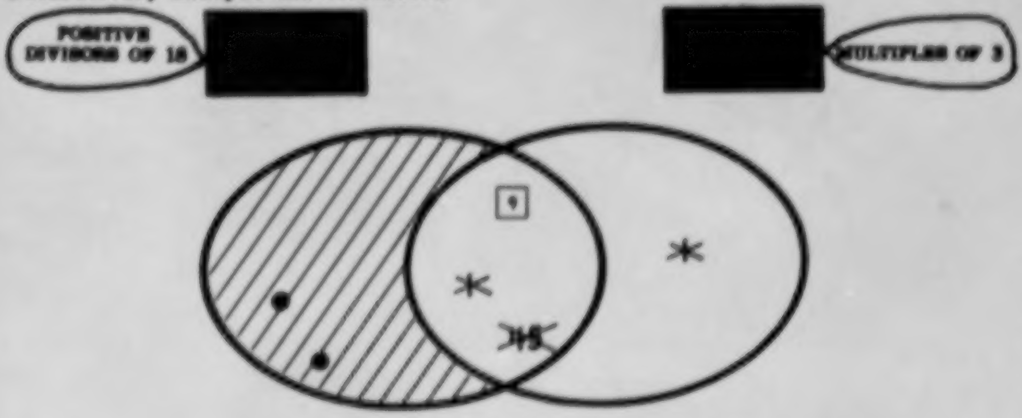
Erase the board and prepare for a two-string analysis activity as shown below.


RED
BLUE
MULTIPLES OF 2
MULTIPLES OF 3
MULTIPLES OF 4
MULTIPLES OF 5
MULTIPLES OF 10
ODD NUMBERS
POSITIVE PRIME NUMBERS
LARGER THAN 20
SMALLER THAN 20
LARGER THAN ⁻10
SMALLER THAN ⁻10
POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27
POSITIVE DIVISORS OF 18
MULTIPLES OF 3
9
15

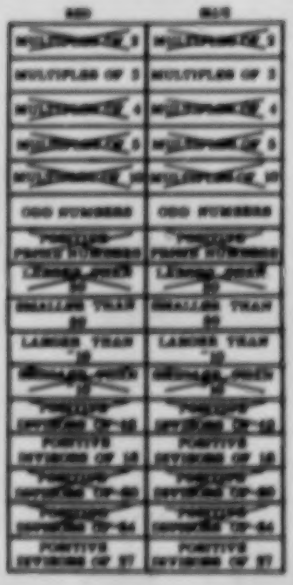
L 7

Distribute sheets from the individual String Game Analysis Pad. Ask the students to cross out labels the strings cannot have and to circle what the strings are for on their lists. If you want to use a game format, give the students a five-to-ten minute time limit in which to discover the string labels. The scoring could be that a team gains five points for each member who identifies only one of the strings correctly and gains fifteen points for each member who identifies both strings correctly.

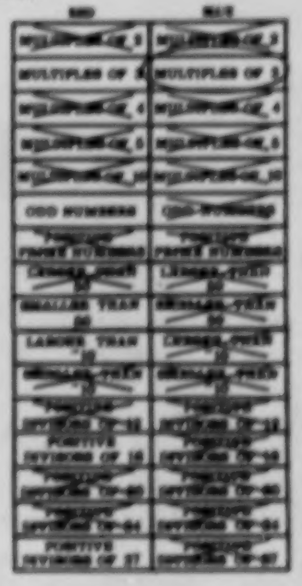
Collectively analyze the situation.

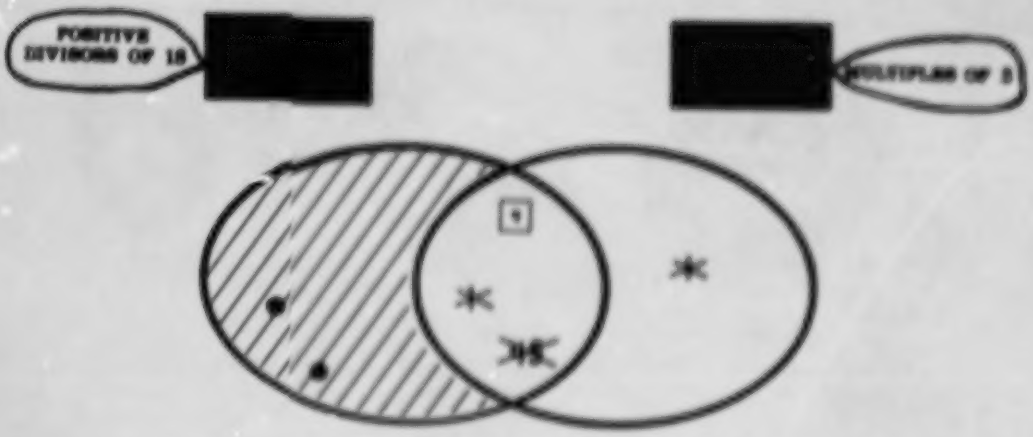


Using the information of where 9 is in the picture, cross off these labels.



Using the information that 1 does not belong in the blue string, cross off additional labels. Decide that the blue string is for the multiples of 3.

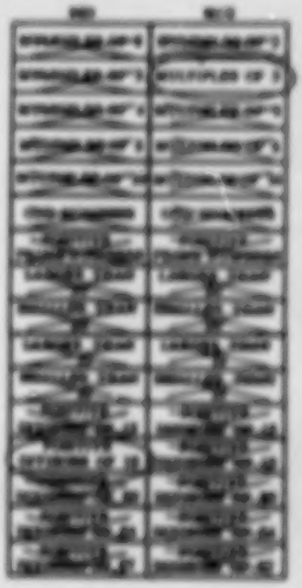



POSITIVE DIVISORS OF 18
MULTIPLES OF 3
9

45 is a multiple of 3 and so must be in the blue string. Using the information that 45 does not belong in the middle region, locate 45 inside the blue string but outside the red string and cross off additional labels.

Using the information that there are exactly two numbers in the hatched region, decide that the red string must be for the positive divisors of 18. (There is only one positive divisor of 27 which is not a multiple of 3.)

RED
BLUE
MULTIPLES OF 3
POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 27


RED
BLUE
MULTIPLES OF 3

L 7

Exercise 2B

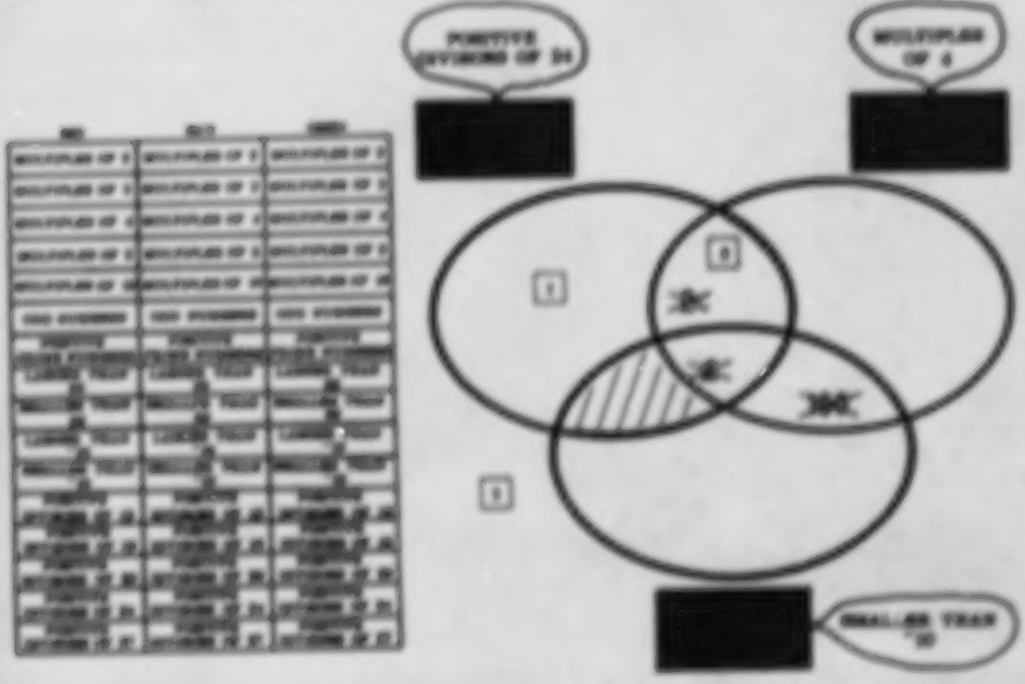
As a group activity, analyse the same situation as in Exercise 2A, except give the clues one at a time.

- Start by locating the number 9 in the picture.
- Secondly, indicate that the number 1 does not belong inside the blue string (in either region).
- Thirdly, indicate that the number 45 does not belong in the middle region.
- Finally, indicate that there are exactly two numbers in the red string which are not in the blue string.

See the description of Exercise 2A for a reference to which labels can be crossed off the lists after each of the above clues is given.

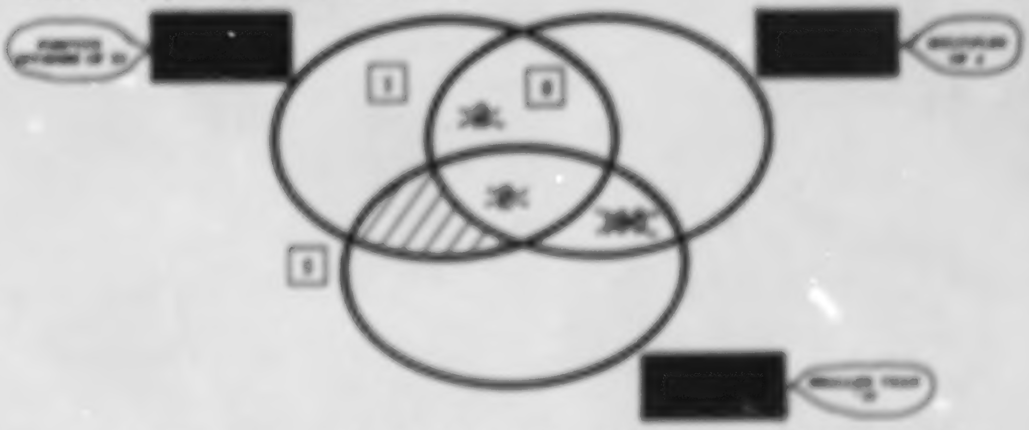
Exercise 3A

Erase the board and prepare for a three-string analysis activity as shown below.

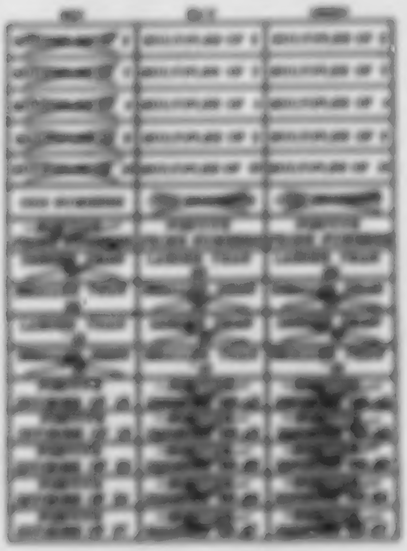

POSITIVE DIVISORS OF 24
MULTIPLES OF 4
RED
BLUE
GREEN
ODD NUMBERS
1
8
9
SMALLER THAN 20

Distribute sheets from the individual String Game Analysis Pad (three-string). Ask the students to cross out labels the strings cannot have and to circle what the strings are for on their lists. If you want to use a game format, give the students a ten-minute time limit in which to discover the string labels. The scoring could be that a team gains five points for each member who identifies only one of the strings correctly, ten points for each member who identifies only two of the strings correctly, and twenty points for each member who identifies all three strings correctly.

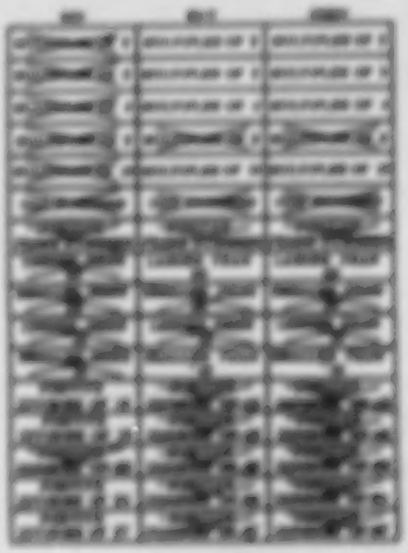
Collectively analyse the situation.



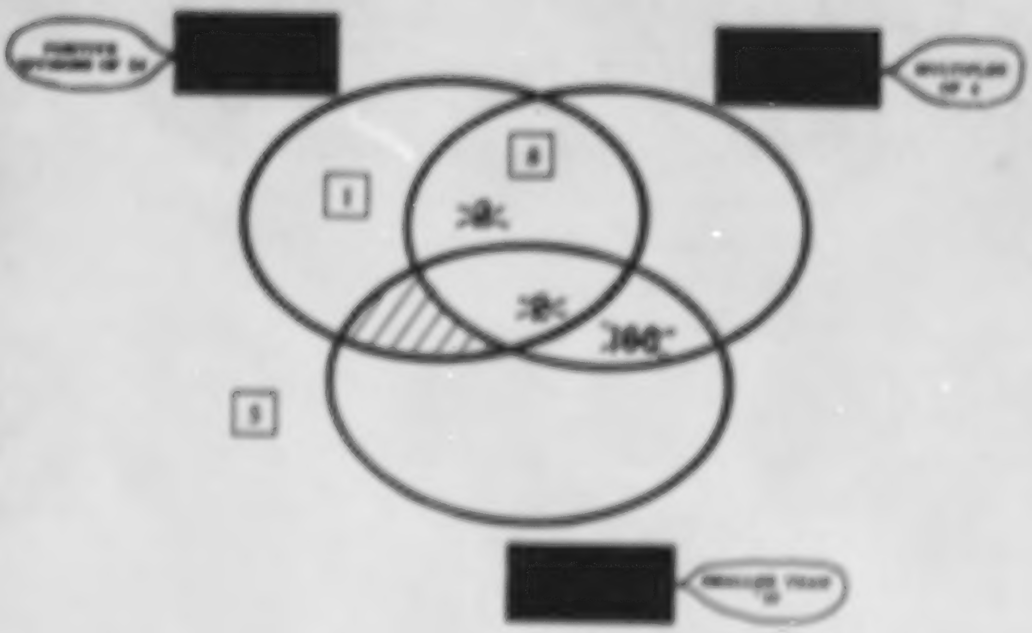
Using the information of where 1 is in the picture, cross off these labels.



Using the information of where 5 is in the picture, cross off additional labels.



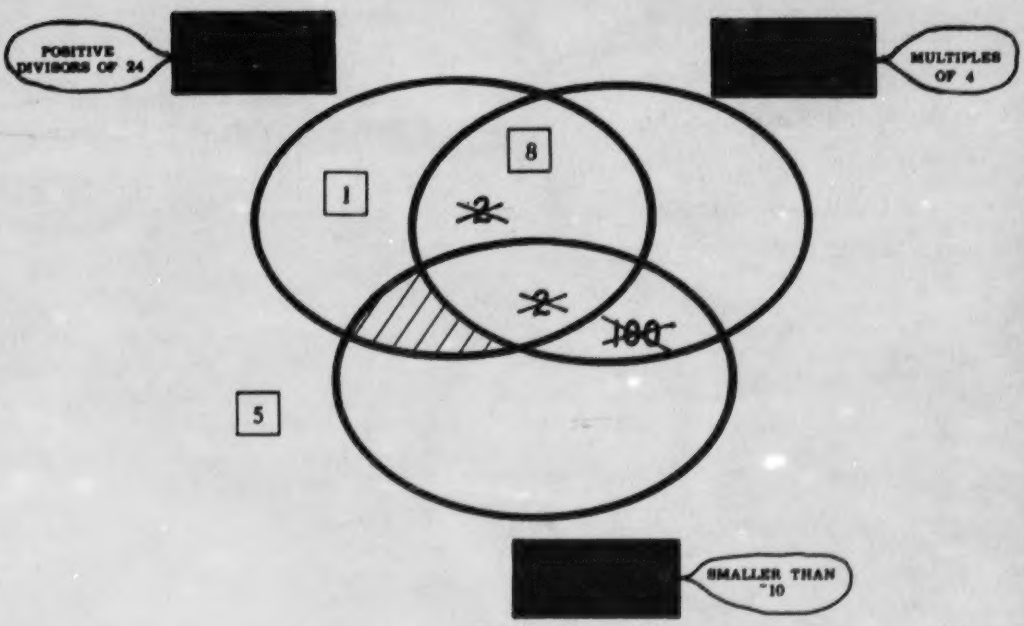
L 7



Using the information of where 8 is in the picture, cross off additional labels. Decide that the red string is for the positive divisors of 16.

1 is a positive divisor of 16 and, therefore, must be in the red string. Using the information about where 1 does not belong, decide that 1 does not belong in the blue string. The blue string must be for the multiples of 4.

L 7



100 is a multiple of 4 and is not a positive divisor of 24. Using the information that 100 does not belong in the indicated region, decide that 100 is not in the green string and cross off additional labels.

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	MULTIPLES OF 3
~~MULTIPLES OF 4~~	[✓ MULTIPLES OF 4]	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN ˉ10~~	~~LARGER THAN ˉ10~~	~~LARGER THAN ˉ10~~
~~SMALLER THAN ˉ10~~	~~SMALLER THAN ˉ10~~	SMALLER THAN ˉ10
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
[✓ POSITIVE DIVISORS OF 24]	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

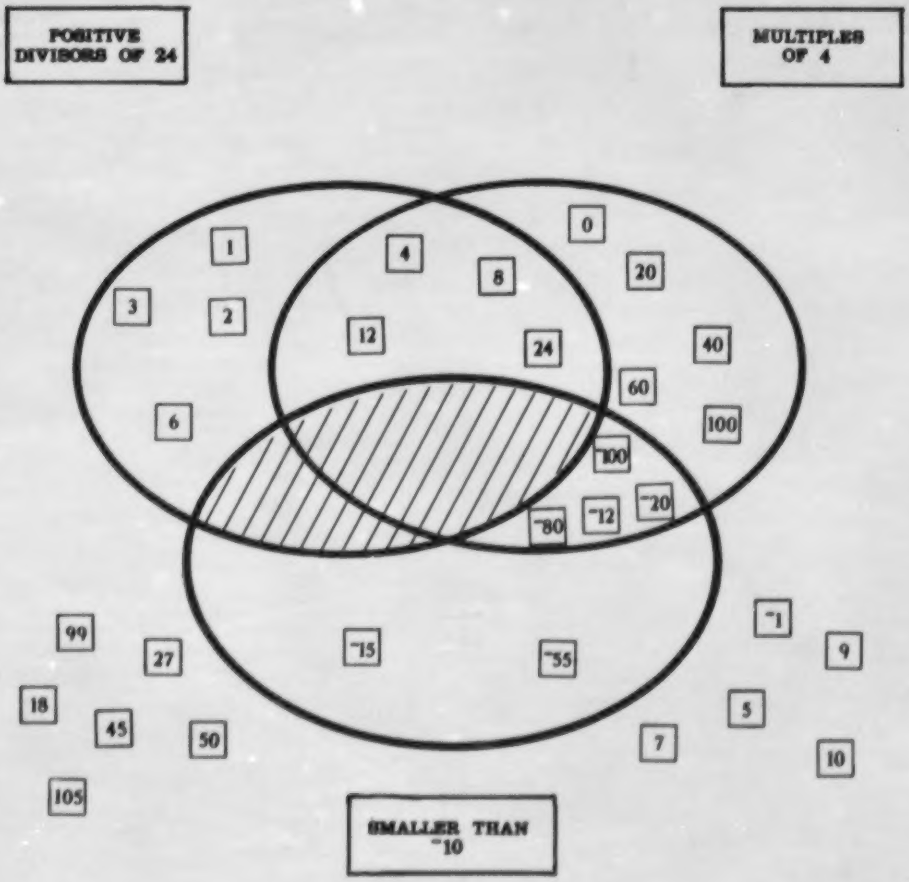
If the green string were for the multiples of 3, the numbers 3 and 6 would be in the hatched region. Conclude that the green string must be for the numbers smaller than ˉ10.

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	[✓ MULTIPLES OF 4]	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~LARGER THAN ˉ10~~	~~LARGER THAN ˉ10~~	~~LARGER THAN ˉ10~~
~~SMALLER THAN ˉ10~~	~~SMALLER THAN ˉ10~~	[✓ SMALLER THAN ˉ10]
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
[✓ POSITIVE DIVISORS OF 24]	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

L 7

Exercise 3B

Play the String Game with three strings, except with the string cards "face-up." The team which is first to place all of its pieces correctly in the picture wins. Do not put any game pieces in the picture yourself. Correct placement of pieces is given below.


POSITIVE DIVISORS OF 24
MULTIPLES OF 4
SMALLER THAN ⁻10
1
3
2
6
4
8
12
24
0
20
40
60
100
⁻100
⁻80
⁻12
⁻20
⁻15
⁻55
99
27
18
45
50
105
⁻1
9
5
7
10

1.5 Games with Strings #5

CAPSULE LESSON SUMMARY

In the context of the Divisors Game, use clues involving the placement of certain numbers to determine the labels of two strings. Do a similar activity in the context of the String Game. Play the String Game with three strings in the usual way or place the game pieces in the picture with the string cards face-up.

MATERIALS

Teacher: Colored chalk; Demonstration (3-string) Analysis Sheet; Numerical String Game kit

Student: Paper; colored pencils; String Game Analysis Pad

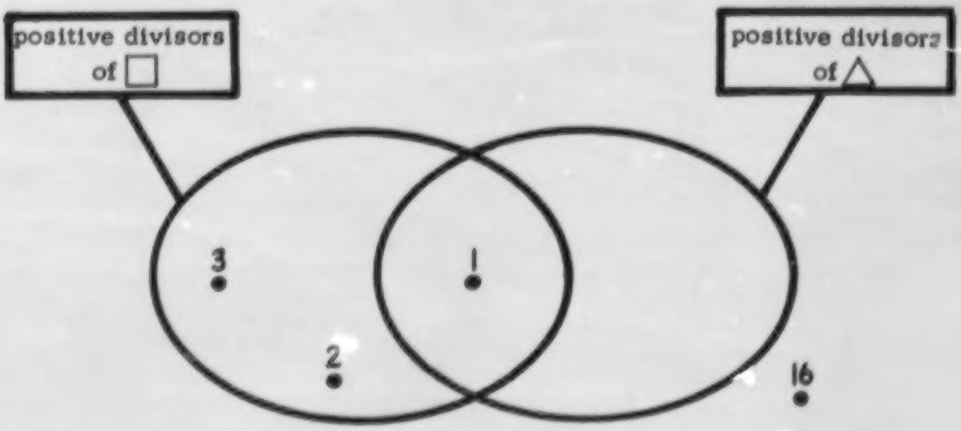
NOTE: The String Game poster for analysis of a 3-string game and the individual 3-string analysis sheets can be used for either a 2-string or 3-string situation. In the former, simply cross out (or fold under) the third column before starting the activity.

DESCRIPTION OF LESSON

There are three exercises in this lesson with two versions of the second exercise, Exercise 2A and Exercise 2B; choose whichever format is more appropriate for your class.

Exercise 1: The Divisors Game

Prepare your board for the Divisors Game. Put the numbers 1, 2, 3, and 16 in the picture as shown below.



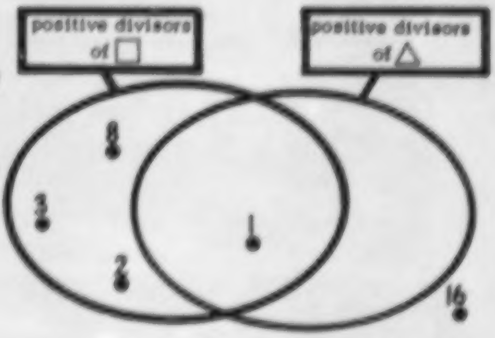
L 9

T: Suppose that we are playing the Divisors Game and that so far these numbers have been located in the picture. Then suppose I stop the game and tell you that the numbers 8 and 11 do not belong outside both strings. Where do they belong? What are the strings for? Remember that in the Divisors Game the numbers which can be in the square and in the triangle are restricted to whole numbers from 1 to 50.

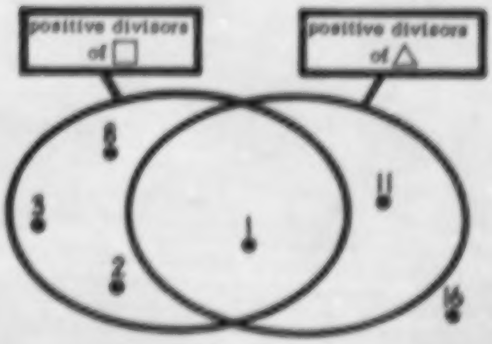
Ask the students to copy the picture on their papers, to locate 8 and 11, and to determine the string labels (there is sufficient information to do this).

Collectively go over the situation. Invite students to draw and label dots in your picture for 8 and 11.

S: 2 must be in the same string(s) that 8 is in. Since we know where 2 belongs, we know also where 8 belongs.



S: 11 cannot be in the red string, otherwise that string would be for the positive divisors of a number larger than 50.[#]



[#] If 11, 2, and 3 were in the same string, that string would be for the positive divisors of a multiple of 66 since 11, 2, and 3 have no common divisors other than 1.

Collectively determine the string labels. An analysis is given below.

- 8 is in the red string so the number in the square must be a multiple of 8 between 1 and 50, namely:

8
16
24
32
40
48

- 16 is outside the red string so the number in the square cannot be a multiple of 16. Cross out "16", "32", and "48" for this reason.

8
~~16~~
24
~~32~~
40
~~48~~

- 3 is in the red string so the number in the square must be a multiple of 3 between 1 and 50. Cross out "8" and "40" for this reason.

~~8~~
~~16~~
24
~~32~~
~~40~~
~~48~~

L 9

- 11 is in the blue string so the number in the triangle must be a multiple of 11 between 1 and 50, namely:

11
22
33
44

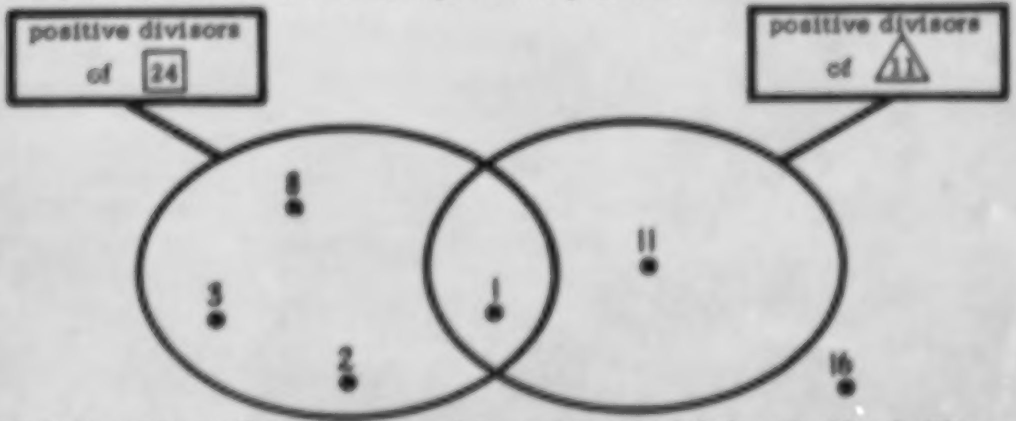
- 2 is not in the blue string so the number in the triangle cannot be a multiple of 2. Cross out "22" and "44" for this reason.

11
~~22~~
33
~~44~~

- 3 is not in the blue string so the number in the triangle cannot be a multiple of 3. Cross out "33" for this reason.

⑪
~~22~~
~~33~~
~~44~~

The picture below shows the strings correctly labeled.

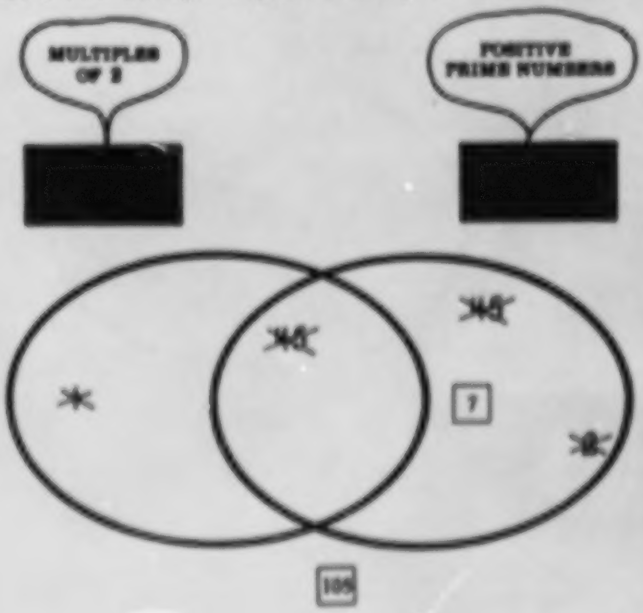


Ask the students if there are any other numbers which belong inside of either string. (24, 12, 4, and 6 belong in the red string but outside the blue string.)

Exercise 2A: The String Game

Erase the board and prepare for a two-string analysis activity as shown below.

RED	BLUE
MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
LARGER THAN 20	LARGER THAN 20
SMALLER THAN 20	SMALLER THAN 20
LARGER THAN ⁻10	LARGER THAN ⁻10
SMALLER THAN ⁻10	SMALLER THAN ⁻10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

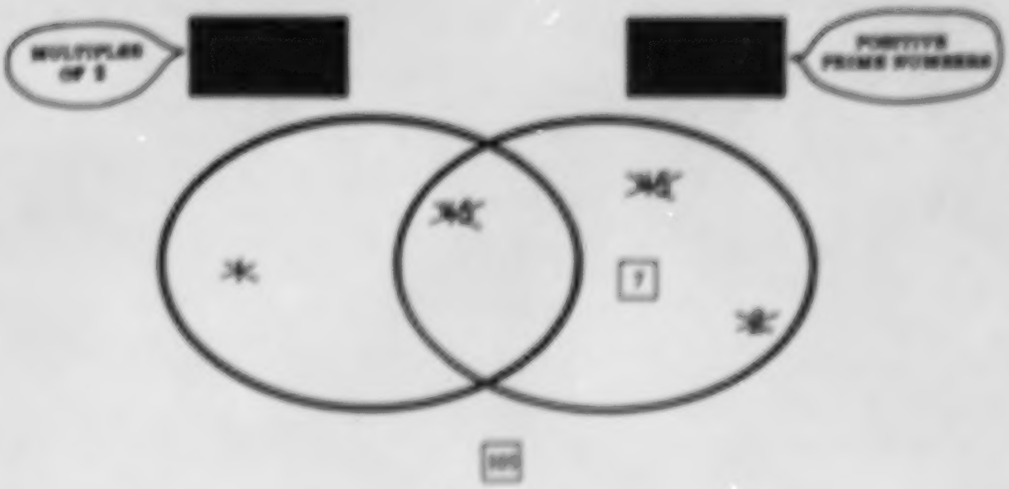


T: It is possible to decide what the strings are *for* with only these number clues.

Distribute sheets from the individual String Game Analysis Pad. Ask the students to cross out labels the strings cannot have and to circle what the strings are for on their lists. If you want to use a game format, give the students a ten minute time limit in which to discover the string labels. The scoring could be that a team gains five points for each member who identifies only one of the strings correctly and gains fifteen points for each member who identifies both strings correctly.

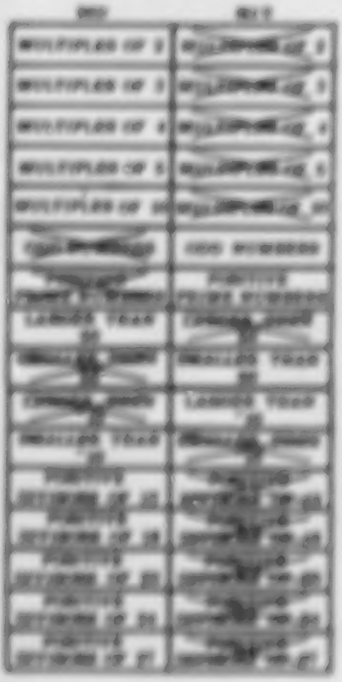
L 9

Collectively analyse the situation.



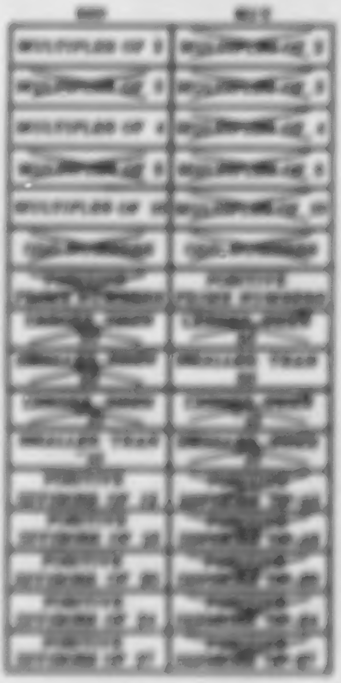
POSITIVE PRIME NUMBERS
7
105

Using the information of where 7 is in the picture, cross off these labels.

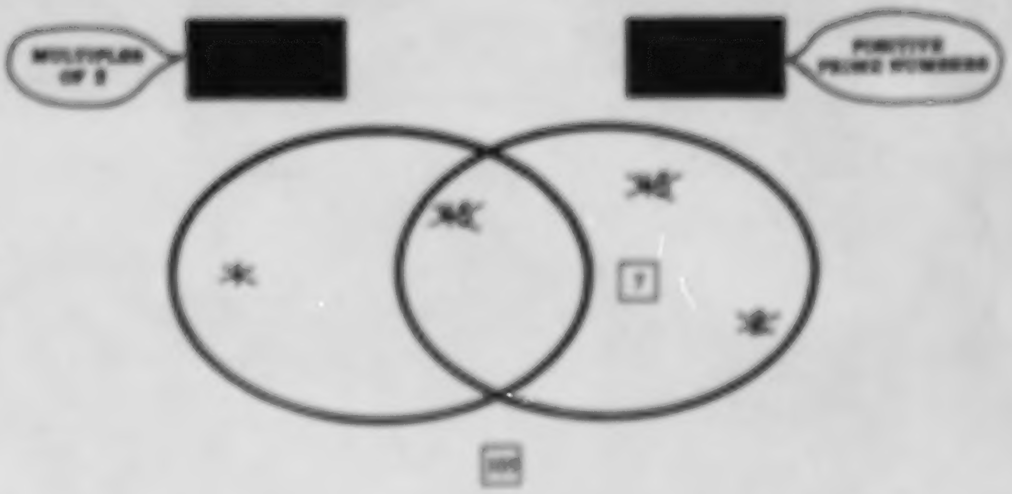


MULTIPLES OF 2
MULTIPLES OF 3
MULTIPLES OF 4
MULTIPLES OF 5
MULTIPLES OF 10
ODD NUMBERS
POSITIVE PRIME NUMBERS

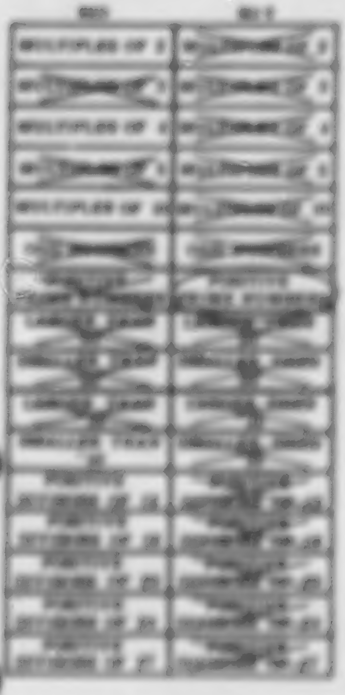
Using the information of where 105 is in the picture, cross off additional labels.



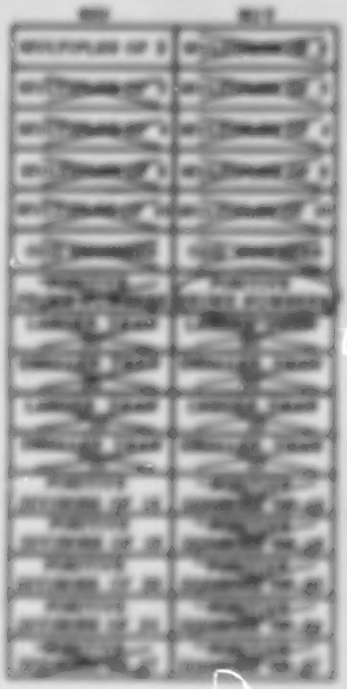
L 9


MULTIPLES OF 2
POSITIVE PRIME NUMBERS
7
45

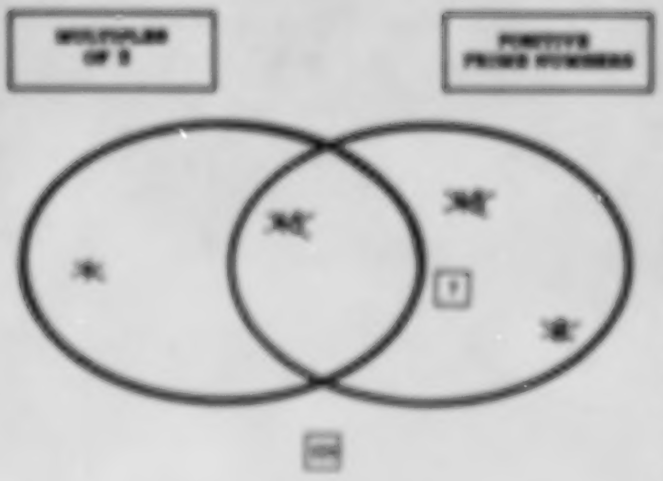
Using the information that 45 does not belong in the blue string, decide that the blue string is not for the numbers smaller than 50. Therefore, the blue string must be for the positive prime numbers.

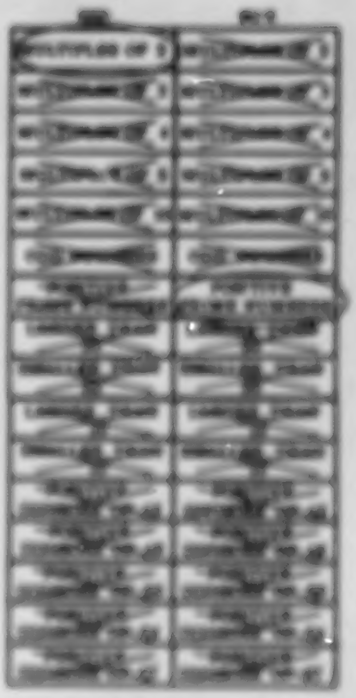

RED
BLUE
MULTIPLES OF 2
MULTIPLES OF 4
MULTIPLES OF 10
POSITIVE PRIME NUMBERS
SMALLER THAN 50
POSITIVE DIVISORS OF 24

2 is a positive prime number and therefore, must be in the blue string. Using the information that 2 does not belong in the indicated region, decide that 2 belongs in the middle region and cross off additional labels.


RED
BLUE
MULTIPLES OF 2
POSITIVE PRIME NUMBERS

L 9





1 is not a prime number so it cannot be in the blue string. Using the information that 1 does not belong in the indicated region, decide that 1 belongs outside both strings. Thus the red string cannot be for the positive divisors of any number; it must be for the multiples of 3.

Exercise 2B: The String Game

As a group activity, analyze the same situation as in Exercise 2A, except give the clues one at a time.

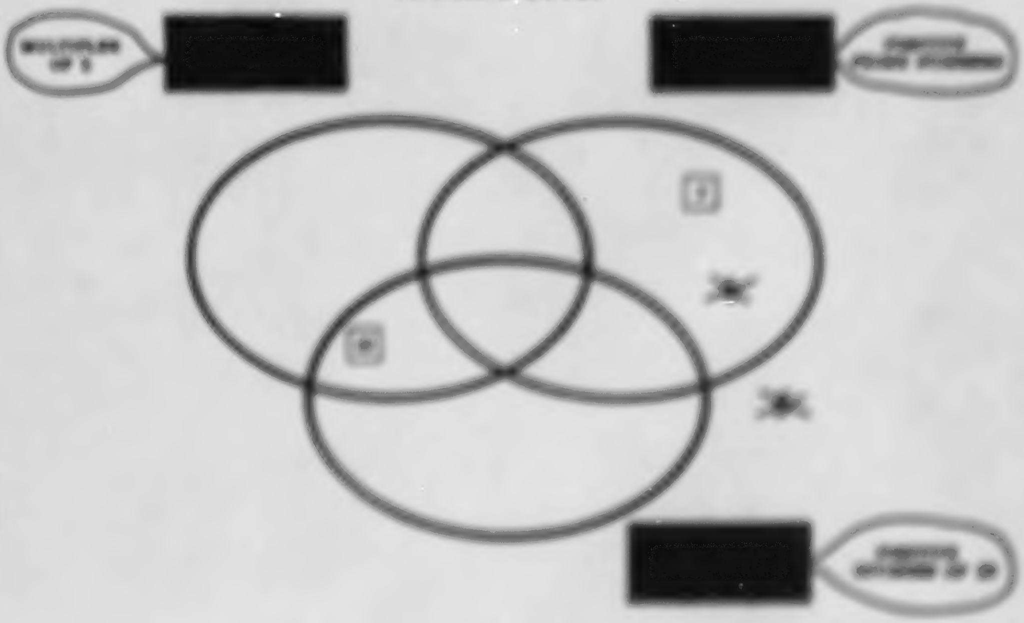
- Start by locating the number 7 in the picture.
- Secondly, locate the number 105 in the picture.
- Thirdly, indicate that the number 45 does not belong in either region of the blue string.
- Fourthly, indicate that the number 3 does not belong in the region of the blue string which is outside the red string.
- Finally, indicate that the number 1 does not belong in the region of the red string which is outside the blue string.

See the description of Exercise 2A for a reference to which labels can be crossed off the lists after each of the above clues is given.

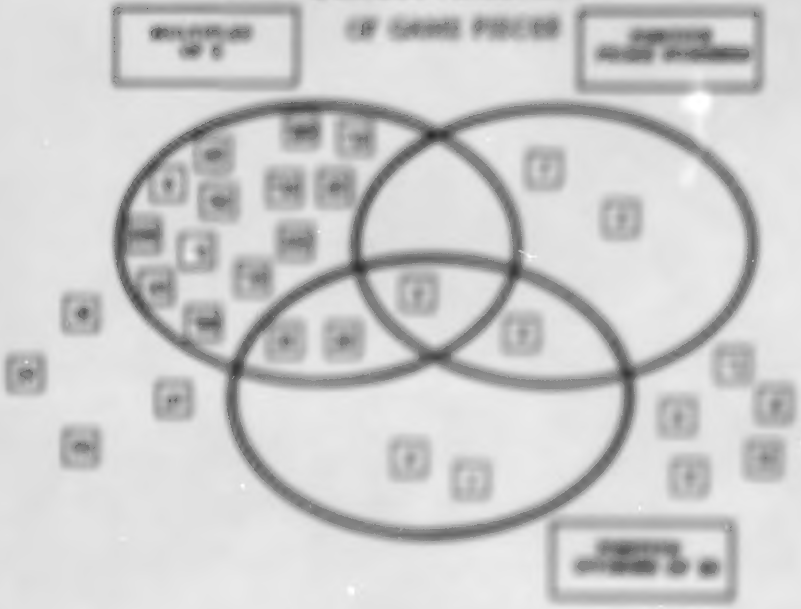
Exercise 3

Play the String Game in the usual way or, if your students need more practice with three string situations, with the string cards face-up. A possible game is suggested below.

STARTING CLUES


MULTIPLES OF 5

CORRECT PLACEMENT OF GAME PIECES


MULTIPLES OF 5

L 11

L4 Games with Strings #1

CAPSULE LESSON SUMMARY

Play the Divisors Game. In the context of the String Game, use clues involving the placement of certain numbers to determine the labels of two strings.

MATERIALS

Teacher: Colored chalk; Demonstration (2-string) Analysis Sheet; Numerical String Game kit

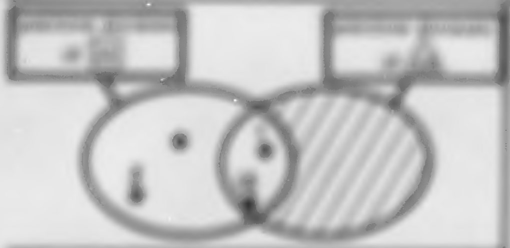
Student: String Game Analysis Pad; Worksheets L11 and L12

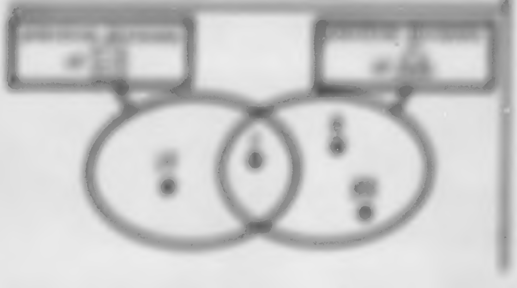
DESCRIPTION OF LESSON

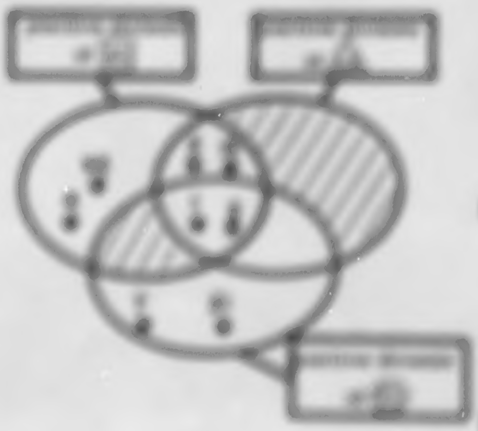
There are two exercises plus worksheets for this lesson. Two versions of the second exercise, Exercise 2A and 2B, are provided; choose whichever format is more appropriate for your class.

Exercise 1: The Divisors Game

Divide the class into two or three teams. Play the Divisors Game twice. Partial info sheets are given below for three possible games. Choose two of these or make up similar ones. Note that only the numbers inside the strings are shown; assume that any whole number from 1 to 50 not shown in the picture is outside the strings. The labeling is included for your reference and is not to be given as a clue. Remember that the only number put in the picture at the beginning of a game is the number 1.

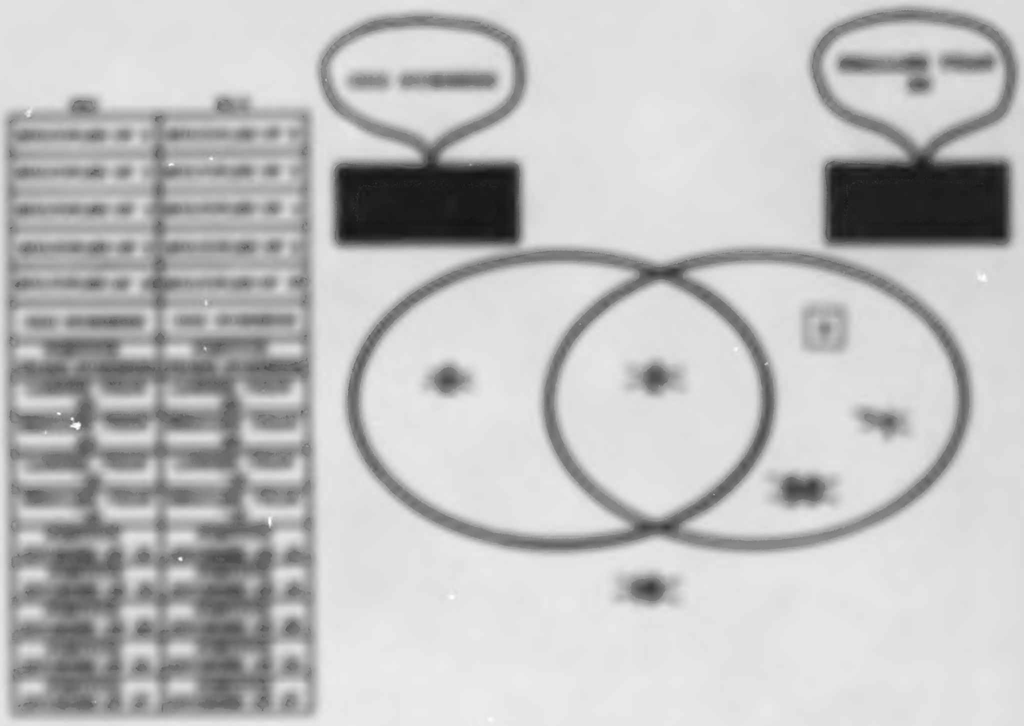






Exercise 23: The String Game

Erase the board and prepare for a two-string analysis activity as shown below.

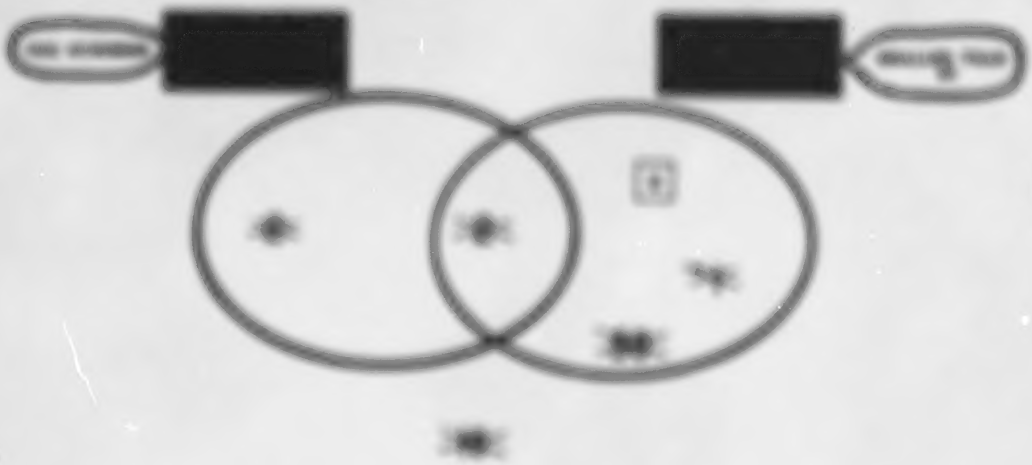


T: It is possible to decide what the strings are for with only these number clues.

Distribute sheets from the Individual String Game Analysis Pad. Ask the students to cross out labels the strings cannot have and to circle what the strings are for on their lists. If you want to use a game format, give the students a ten minute time limit in which to discover the string labels. The scoring could be that a team gains five points for each member who identifies only one of the strings correctly and gains fifteen points for each member who identifies both strings correctly.

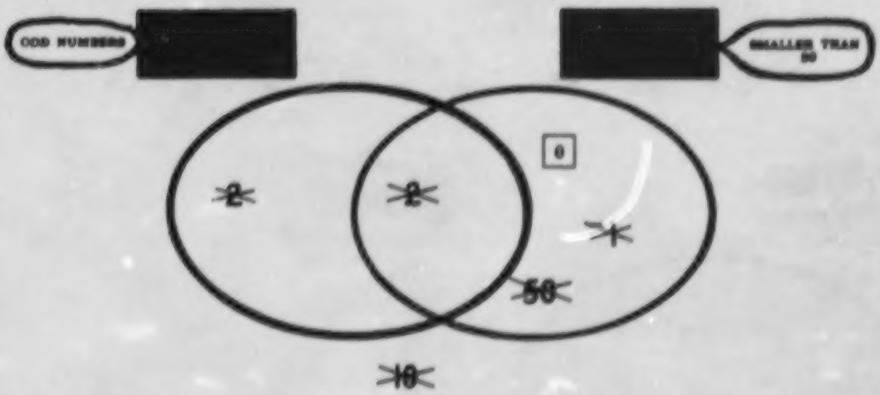
L 11

Collectively analyze the situation.



Using the information of where X is in the picture, cross off those letters.

Using the information that X does not belong in either region of the red string, cross off additional letters.



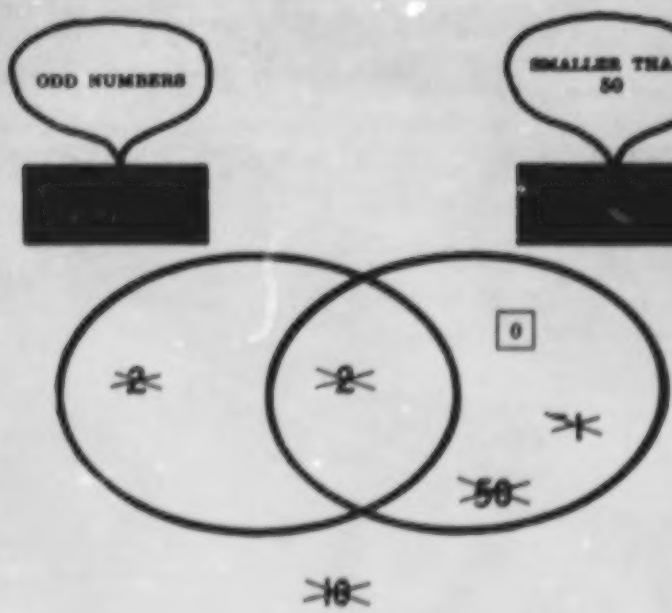
Considering the four possibilities left for the red string, decide that 10 cannot be in the red string. Using the information that 10 does not belong outside both strings, decide that 10 is in the blue string and cross off additional labels.

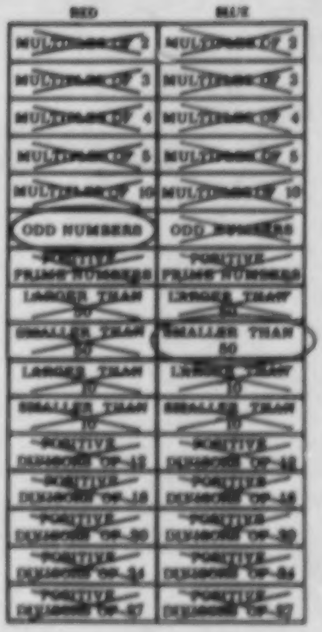
RED	BLUE
~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	MULTIPLES OF 10
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
LARGER THAN 50	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	SMALLER THAN 50
~~LARGER THAN -10~~	LARGER THAN -10
SMALLER THAN -10	~~SMALLER THAN -10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

Considering the four possibilities left for the red string, decide that 50 cannot be in the red string. Using the information that 50 does not belong in the indicated region, decide that 50 is outside both strings and thus determine the blue string is for the numbers smaller than 50.

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
LARGER THAN 50	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	SMALLER THAN 50
~~LARGER THAN -10~~	~~LARGER THAN -10~~
SMALLER THAN -10	~~SMALLER THAN -10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

L 11





RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~LARGER THAN 50~~	~~LARGER THAN 50~~
~~SMALLER THAN 50~~	SMALLER THAN 50
~~LARGER THAN 10~~	~~LARGER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

¯1 is smaller than 50, so ¯1 is in the blue string. Using the information that ¯1 does not belong in the indicated region, decide that ¯1 is in the middle region. Therefore, the red string must be for the odd numbers.

__Exercise 2B__: The String Game

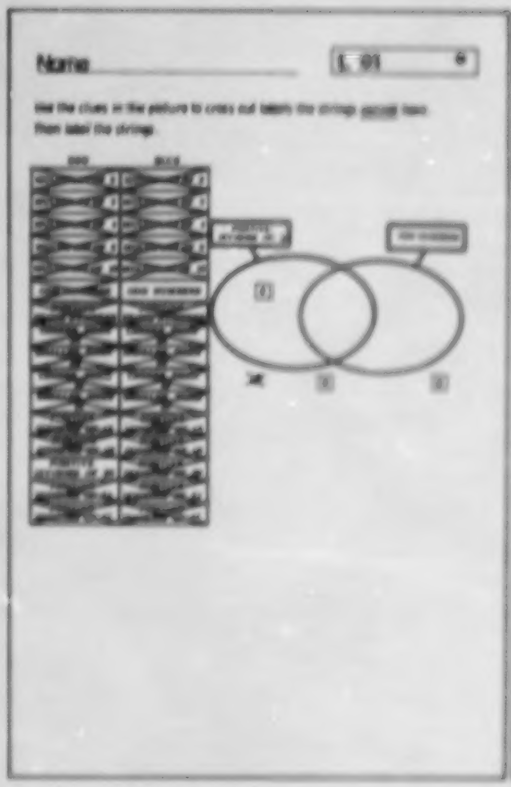
As a group activity, analyze the same situation as in Exercise 2A, except give the clues one at a time.

- Start by locating the number 0 in the picture.
- Secondly, indicate that the number 2 does not belong in either region of the red string.
- Thirdly, indicate that the number 10 does not belong outside the strings.

- Fourthly, indicate that the number 50 does not belong in the region of the blue string which is outside the red string.
- Finally, indicate that the number ⁻1 does not belong in the region of the blue string which is outside the red string.

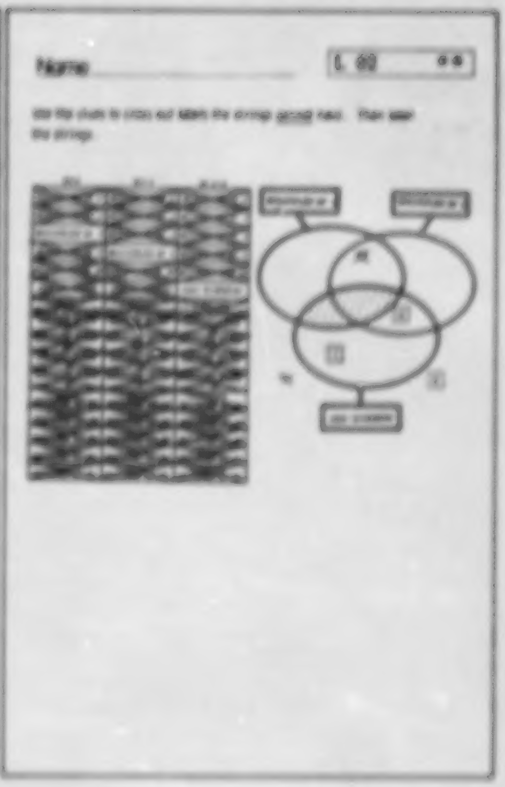
See the description of Exercise 2A for a reference to which labels can be crossed off the lists after each of the above clues is given.

Worksheets L01 and L02 are available for independent work.



Name

L 01



Name

L 02

2

NECKLACES

Often in mathematics we find a model or framework in which to better examine a problem. In the Necklace lessons[*] interesting combinatorics questions are posed and, by using simple arithmetic models, one not only suspects the solutions but is totally convinced of them.

The first of the two Necklace lessons asks how many distinct necklaces can be made with exactly three red beads and seven white beads. Experimentation leads in a very natural way to a simple coding technique and through arithmetic observations the solution becomes evident. The problem is extended to find how many ways the beads could be put on a pole with one end affixed to the ground. Students discover that the number of possibilities increases.

The second lesson poses a variation of the question: How many distinct necklaces can be made with exactly _four_ red beads and _six_ white beads? The experimentation is similar, the coding is related, but the arithmetic considerations are far more complex.

CAPSULE LESSON SUMMARIES

Section 2.1:	Determine how many ways there are to arrange seven white beads and three red beads in a necklace. Then determine how many ways there are to arrange them on a straight pole. (Worksheets L03, L04)
Section 2.2:	Determine how many ways there are to arrange six white beads and four red beads in a necklace. Then determine how many ways there are to arrange them on a pole. (Worksheets L05, L06, L07)

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[*] It was at the Second CSMP International Conference on the Teaching of Geometry at the Pre-College Level (March 19-29, 1970), that Arthur Engel suggested the main activity of these lessons.

L 3

2.1 Necklaces #1

CAPSULE LESSON SUMMARY

Determine how many ways there are to arrange seven white beads and three red beads in a necklace. Then determine how many ways there are to arrange them on a straight pole.

MATERIALS

Teacher: Two copies of Language Poster #1 for IG-V; colored chalk; tape; red felt-tip markers

Student: Red pencil; Worksheets L03 and L04

NOTE: Before the lesson begins, cut out the pictures of necklaces from both copies of Language Poster #1.

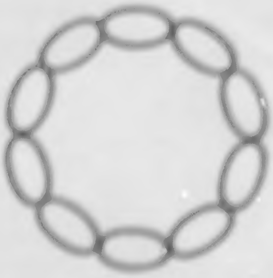
DESCRIPTION OF LESSON

There are two exercises in this lesson. If most of your students are average or better, you will very likely be able to cover the material in both exercises in a 40 to 50 minute class period. Otherwise, aim to cover only the first exercise of this lesson.

Exercise 1

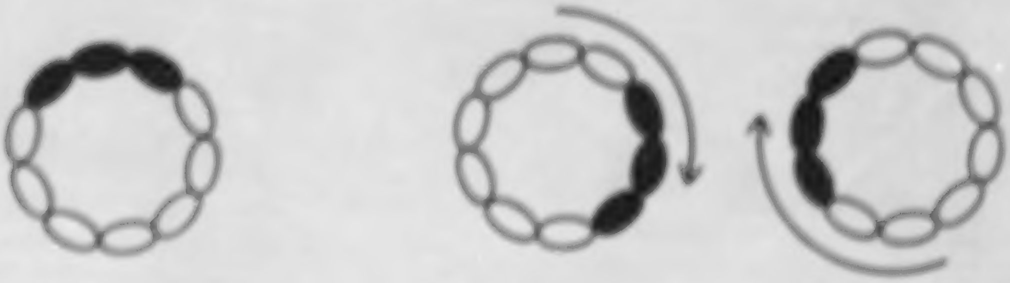
T: Theophilus is a boy who lives in a village where each young person wears a ring of ten beads around the neck. The ring has all white beads when the person receives it.

Tape one picture of a necklace to the board.



T: A white bead is traded for a red bead when the person has performed a special task or has demonstrated a certain skill. Theophilus has done well and has earned three red beads for his necklace. He wonders how to arrange the beads; he could put the three red beads together or space them in the necklace.

Color three beads red in the picture so that they are together. Take the picture off the board and rotate it several times as you say:



T: This describes the same necklace as this one . . . and this one.

Tape the picture back to the board.

T: How many different arrangements of the beads do you think there are?

Record the estimates of some of the students.

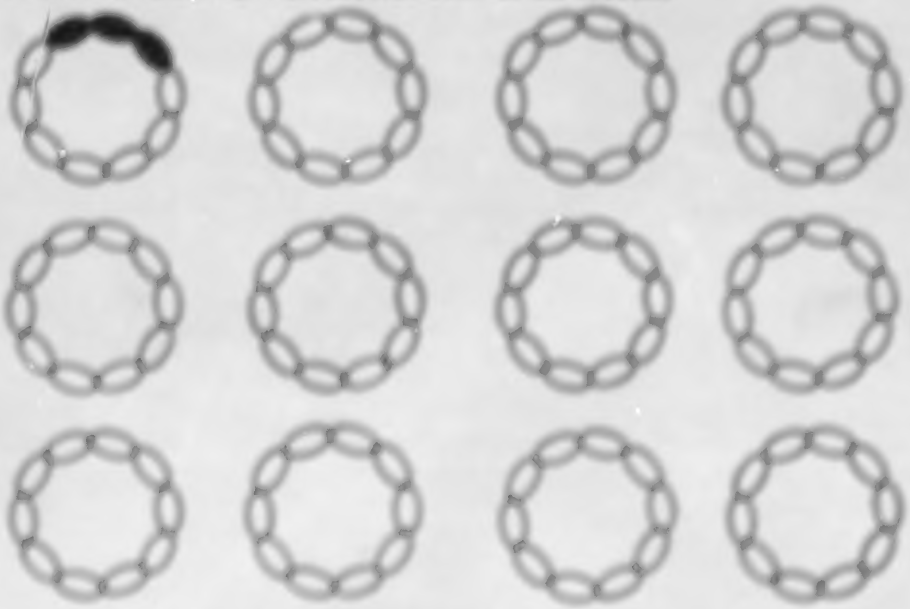
T (holding a copy of Worksheet L03): By coloring three beads red in several pictures, show the different ways Theophilus could arrange the beads in his necklace.

Distribute copies of Worksheet L03 and red pencils.

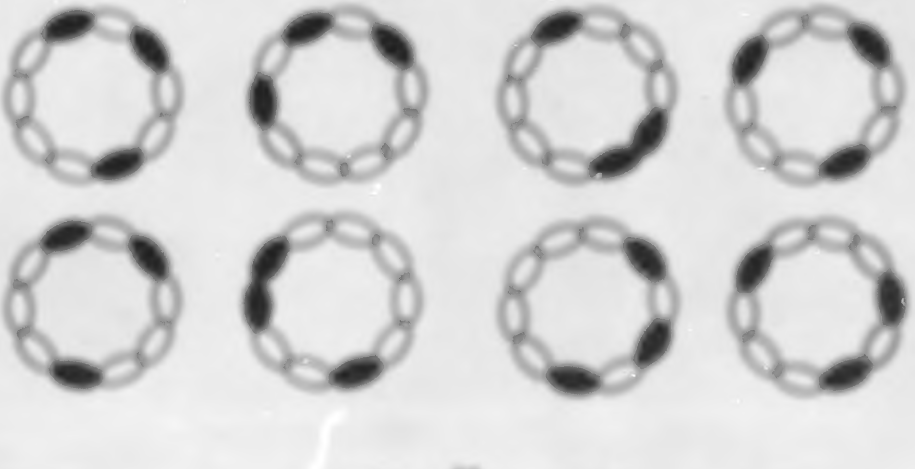
NOTE: There are more than enough pictures on the worksheet to show the complete set of possibilities, but do not tell this to the students.

L 3

Let the students work independently for about ten to fifteen minutes. As they are working, tape eleven more pictures of the necklace to the board. Leave space between them for a discussion later in the lesson.



Afterward invite students to color beads in the pictures on the board. Students should use a red marker to color the beads so the color will show through to the other side of the paper. Your board might look like this.



T: Do any of these colorings describe the same necklace?

If beads are colored in more than eight pictures, there will surely be some duplication. When someone claims that a pair of colorings are duplicates, take one of them off the board. Holding it in the same position as it was on the board, ask:

T: Does this describe the same necklace as this one (point to the suggested duplicate)?

S: Yes, just turn it.

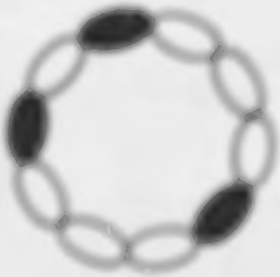
Rotate the picture and let the class see if the beads can be shown to be in the same position. If one picture is not simply a rotation of the other, ask:

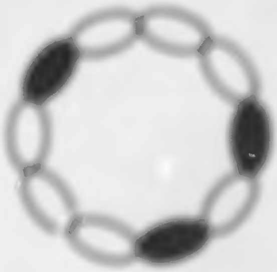
T: Some of you think these describe the same necklace, but just turning one of them does not show that. Is there something else we could do?

S: Turn the picture over.

If no one suggests flipping the picture, suggest it yourself.

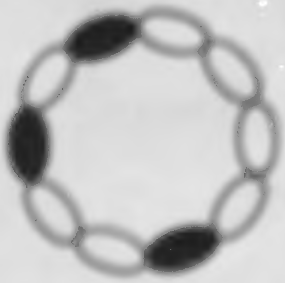
In the preceding illustration, these are rotations of one another. . .

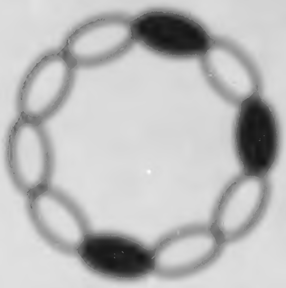




L 3

. . . and these flips of one another.





NOTE: In some cases, you may need to both flip and rotate a picture to show that it is a duplicate of another picture.

Continue in this manner until all duplicates are found. Take one of each pair of duplicates off the board.

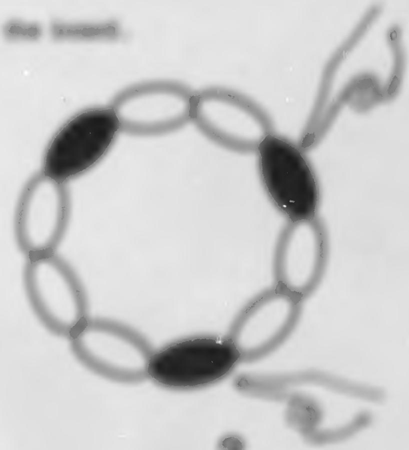
T: How do we find duplicates? What do we look at?

S: I check the number of white beads between pairs of red beads.

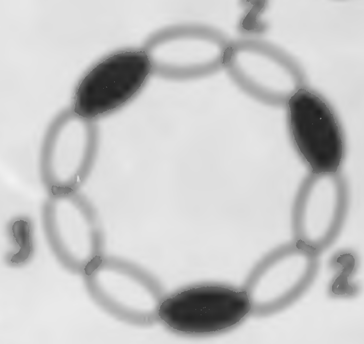
Consider any one of the necklaces pictured on the board.

T: How many white beads are between these two red beads?

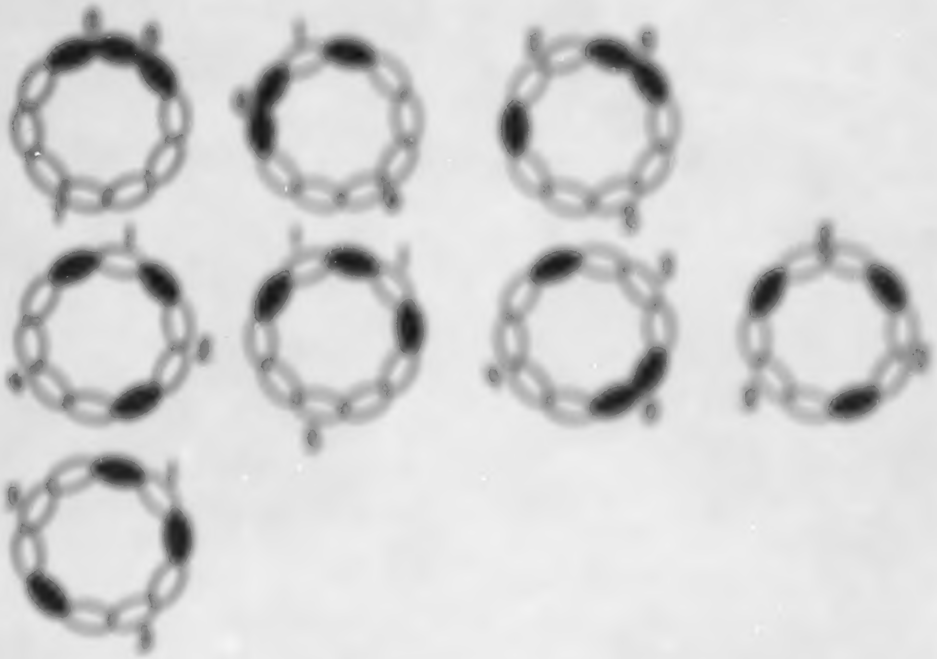
S: Two.



Write "2" near those white beads. Similarly consider the other groups of white beads in that necklace.



Invite some students to record the number of white beads in each group for each necklace pictured on the board.



T: What do you notice about these numbers?

S: There are three numbers for each necklace.

S: The sum of the three numbers for any necklace is 7.

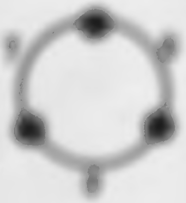
T: How do we know whether or not we have found all the possibilities? What could we do to check?

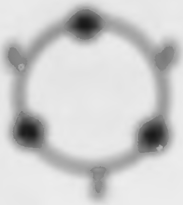
S: Look at the numbers.

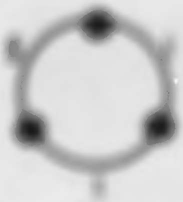
T: We know there are three numbers and their sum is 7. On a piece of paper, write down all the possible combinations of three (whole) numbers whose sum is 7.

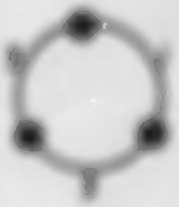
L 3

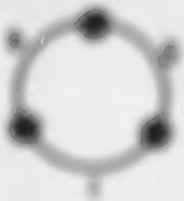
After a few minutes ask the students to give the combinations found. For each one, draw a circle with three red dots and record the numbers in a combination between the dots. Make sure your students understand that each dot is for a red bead and the number between two dots is the number of white beads between the corresponding two red beads.

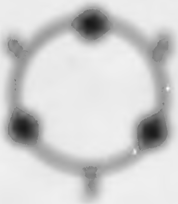


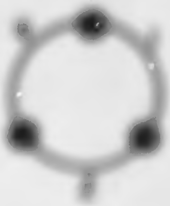


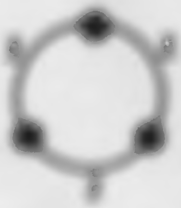












Compare the possible number combinations to those recorded by the necklace pictures. Conclude that the eight distinct necklaces have been described, or use the possible number combinations to find necklaces which are missing. (Some pictures of the necklaces are included on the posters.)

Take the pictures off the board and record on the side of the board the number of solutions. Compare the actual number of ways (eight) to your students' estimates.

necklace 8 ways

Session 2

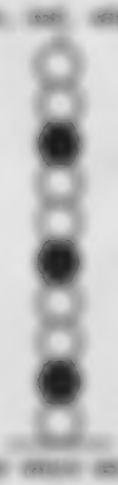
6. Theophilus has a friend in another village. A similar thing is done for the young people in that village. The beads are put on a pole, and the pole is stuck in the ground in front of the young person's house. Just as Theophilus, his friend has earned three red beads.

On the board draw a picture of a pole with ten beads.

T: Who can suggest one way of arranging the beads?

As a student gives the color of the beads, [illegible] color the picture on the board.

S: White, red, white, white, red, white, white, red, white, white.



T: We need to find how many ways we can arrange the three red and seven white beads on a pole. Do you think there are more or less than eight? Why?

S: I think there are more than eight, because if you put the beads in one way and turn it upside down, you have a different arrangement.

L3

S: More than eight, because we can't change the pole like we can the necklace.

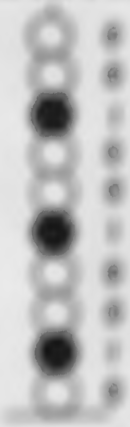
T: Can you suggest a method we could use to count the various arrangements?

S: We could code the arrangements: 0 for a white bead and 1 for a red bead.

Accept this as a good method and then point to the picture on the board as you ask:

T: What is the codeword for this arrangement?

If no one suggests this code, begin writing the codeword for whichever arrangement is pictured on the board and ask a student to finish writing it.



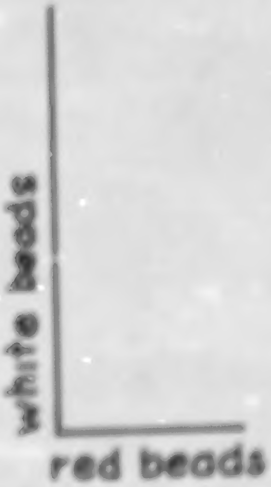
T: The codewords we are concerned with have exactly three 1's for red beads and exactly seven 0's for white beads. Where have we looked at codewords like these before?

S: In THE HIDDEN TREASURE situations.

Students may also recall other situations in the program where binary "words" were used, but THE HIDDEN TREASURE is a very similar situation and should be mentioned. Ask the students to tell the story of THE HIDDEN TREASURE: Eric, a detective, finds all the ways to get from the house (H) to a place where a stolen treasure is hidden (T).

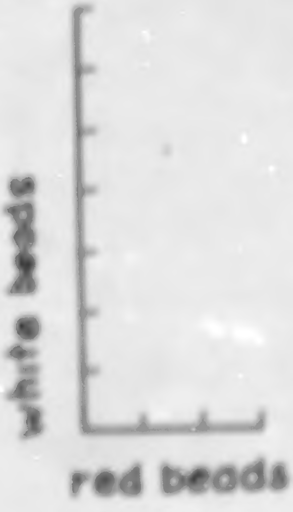
T: Here we can use Spick's method to count the number of arrangements.

Draw this picture on the board as you suggest:


white beads
red beads

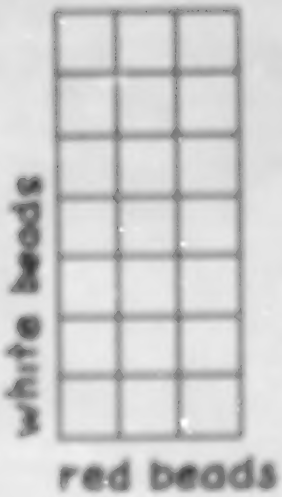
T: Let's represent red beads along the horizontal segment and white beads along the vertical segment.
How many white beads are in Theophilus' necklace? (Seven)
How many red beads . . . ? (Three)

Draw the appropriate graduations on the picture.

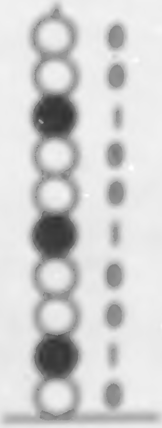

white beads
red beads

L 3

Complete the grid.


white beads
red beads

On the grid, color horizontal segments red for red beads and vertical segments white (black is used here) for white beads to correspond to the arrangement pictured on the board. Use the side of the chalk to make wide, clearly visible marks. Explain to the class what you are doing as you do it.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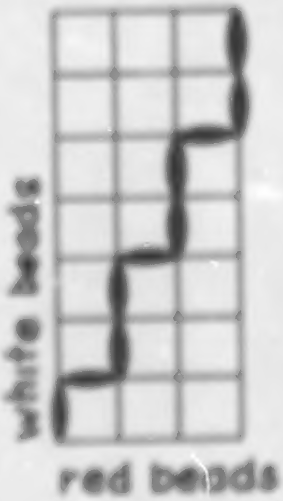

white beads
red beads

T: Who would like to trace an arrangement on the grid as I describe it?

Choose any arrangement of three red and seven white beads (other than the one pictured on the board) and let the volunteer trace it.

T: For any arrangement of three red and seven white beads, where do we start on the grid?

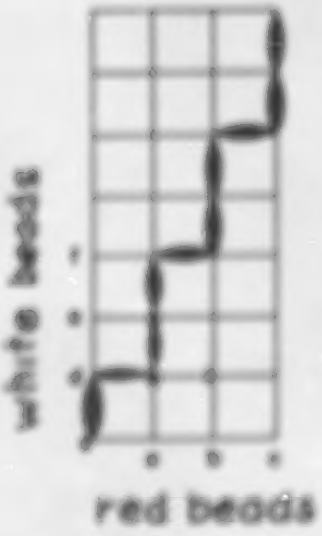
S: At the lower left corner.

T: Where do we end?

S: At the upper right corner.

T: So in this setting we are asking how many paths there are from here (point to lower left corner) to here (point to upper right corner). Do you remember the method Spick used to count such paths?

Let students comment and then begin writing some of the numbers on the grid.



Do not write the letters on the board. They are [illegible] [illegible] [illegible] [illegible] the description of the lesson easier to follow.

L 3

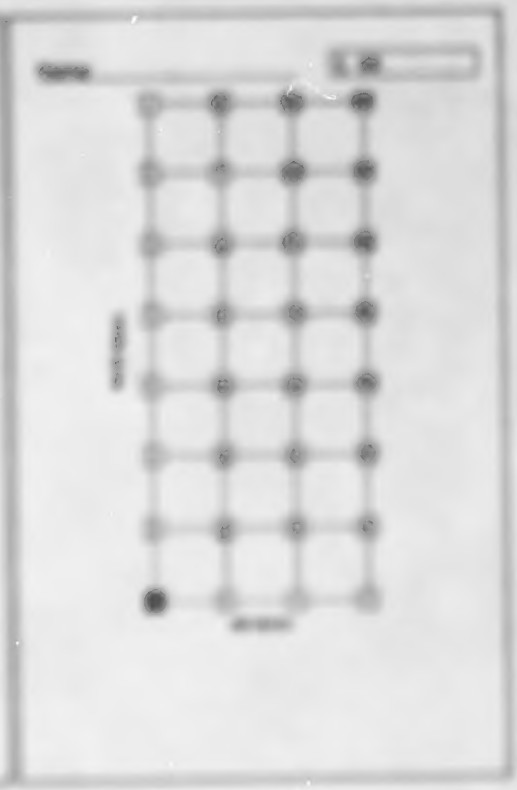
With the class decide:

- there is only one way to go from a to each of the points a, b, c, d, e, and f;
- there are two ways to go from a to h (one through d and one through e)
- there are three ways to go from a to g (one through e and two through h)
- there are three ways to go from a to i (one through h and one through k)

Distribute copies of Worksheet 124 and ask the students to fill in all the numbers on the grid.

Collectively conclude that there are one hundred twenty possible arrangements of three red and seven white beads on a pole.

NECKLACE	8 ways
POLE	120 ways



L 4

L4 Necklaces #1

CAPSULE LESSON SUMMARY

Determine how many ways there are to arrange six white beads and four red beads in a necklace. Then determine how many ways there are to arrange them on a pole.

MATERIALS

Teacher: Two copies of Language Poster #1 (in IG-V); colored chalk; tape; red felt-tip markers

Student: Red pencil; Worksheets 125, 126, and 127

NOTE: Before the lesson begins, cut out the pictures of necklaces from both copies of Language Poster #1.

DESCRIPTION OF LESSON

If your class did not cover the material in Exercise 2 of Lesson P3.1, do only Exercise 1 of this lesson.

Exercise 1

Ask the class to recall the story of Theophilus.

T: Do you remember how many distinct ways there are to arrange three red and seven white beads in a necklace?

S: Eight.

Record this information on the board.

NECKLACE

3 red - 7 white | 8 ways

T: Theophilus has earned a fourth red bead for his necklace and now must decide how to arrange the four red beads and six white beads. Today we are going to find the number of ways for Theophilus to do this. Do you think there are more than or less than eight?

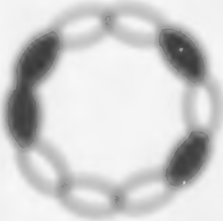
NECKLACE

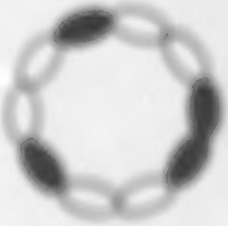
3 red - 7 white	8 ways
4 red - 6 white	?

Perhaps you will have some students who argue that there are more than eight and some students who argue that there are less than eight. Let students give estimates and record them on the board. These estimates will be referred to later in the lesson.

T: I want you to try to solve the problem on your own. Use any method you like to solve it.

Distribute copies of Worksheet L4 and red pencils. Students may want to have paper in addition. Let the students work for about five to ten minutes and then draw the class' attention back together to talk about the problem. Ask two students each to color one picture of a possible necklace at the board. Suppose these colorings are given.



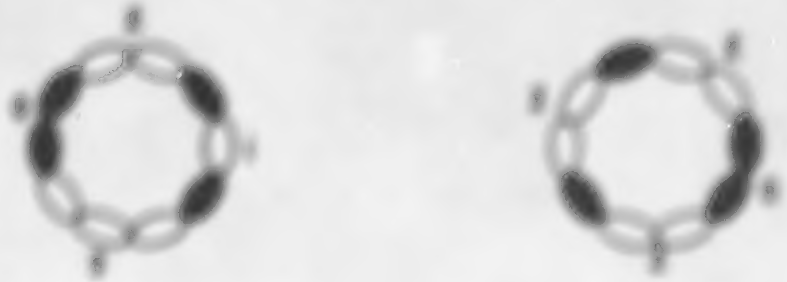


T: Are these pictures of the same necklace or not? How can you tell?

S: They are different; the one on the left has a group of two white beads, a single white bead, and a group of three white beads. The one on the right has three groups of two white beads.

L 4

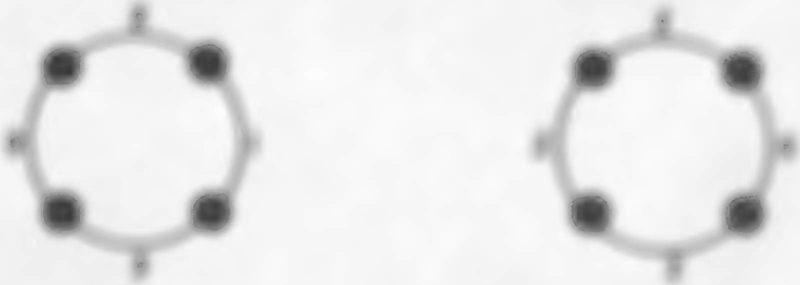
As a student explains, record the number of white beads in each group as in the next illustration. Emphasize that "0" is recorded when two red beads are next to one another.



T: What is the sum of the numbers in each case?

S: 6, because there are six white beads.

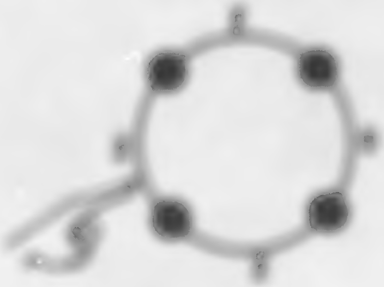
Represent each combination of numbers in a circle code as shown below.



T (holding a copy of Worksheet [illegible]): On this sheet, use it to record here any arrangements you have found and then try to use the code to find other arrangements.

Let the students work individually. Occasionally encourage the class by commenting on the number of arrangements one of the students has found. After about ten minutes, discuss the various arrangements collectively.

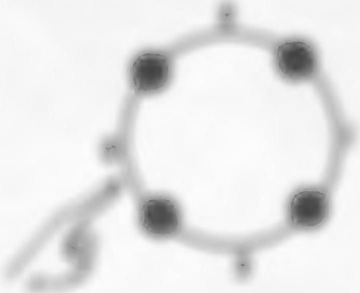
Place one of the two circle codes described earlier on the board. Here is a sample dialogue.



T: Is there a different arrangement of the beads which has the same numbers as this in the circle code?

S: No, you would need only to turn the picture to see that they are the same necklace.

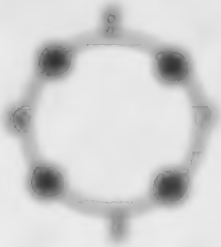
T: What about this one?

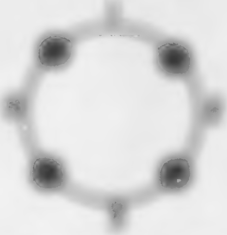


Let students discuss and experiment with this problem for a moment. Eventually they should recognize a different arrangement with the same numbers.

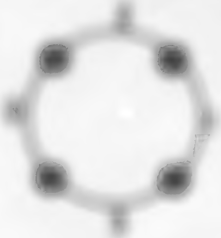
L 4

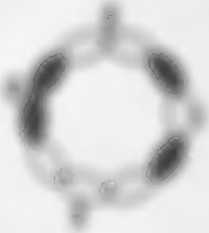
[illegible] Write "[illegible]" on the top, "[illegible]" on the right, and keep "[illegible]" and "[illegible]" in the same positions.

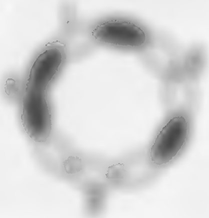




Tape a picture of the necklace on the board and invite a student to color the pictures according to the new circle code.







Lift one of the pictures and compare it to the other.

? (rotating and flipping the picture): Are these pictures for the same arrangement? Can I turn this to make the pictures look the same? Can I flip it and turn it to make the pictures look the same?

[illegible] Etc.

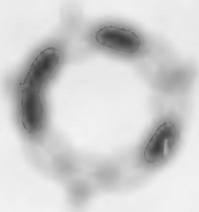
[illegible]

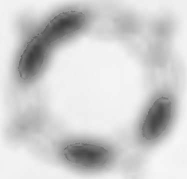
T: [illegible]

[illegible]

S: [illegible]

[illegible]





[illegible]

[illegible] [illegible]

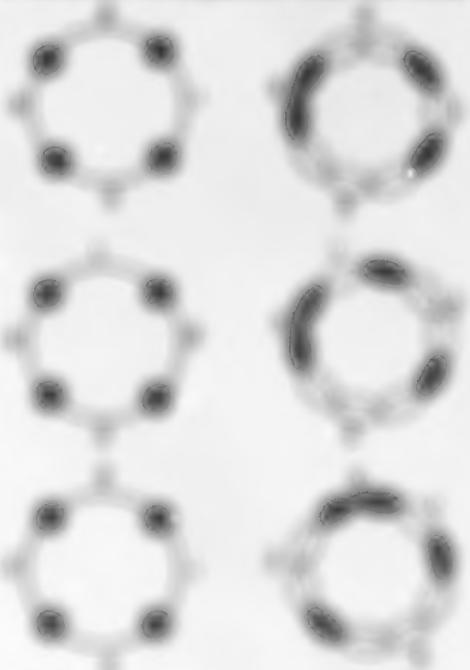
[illegible] [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

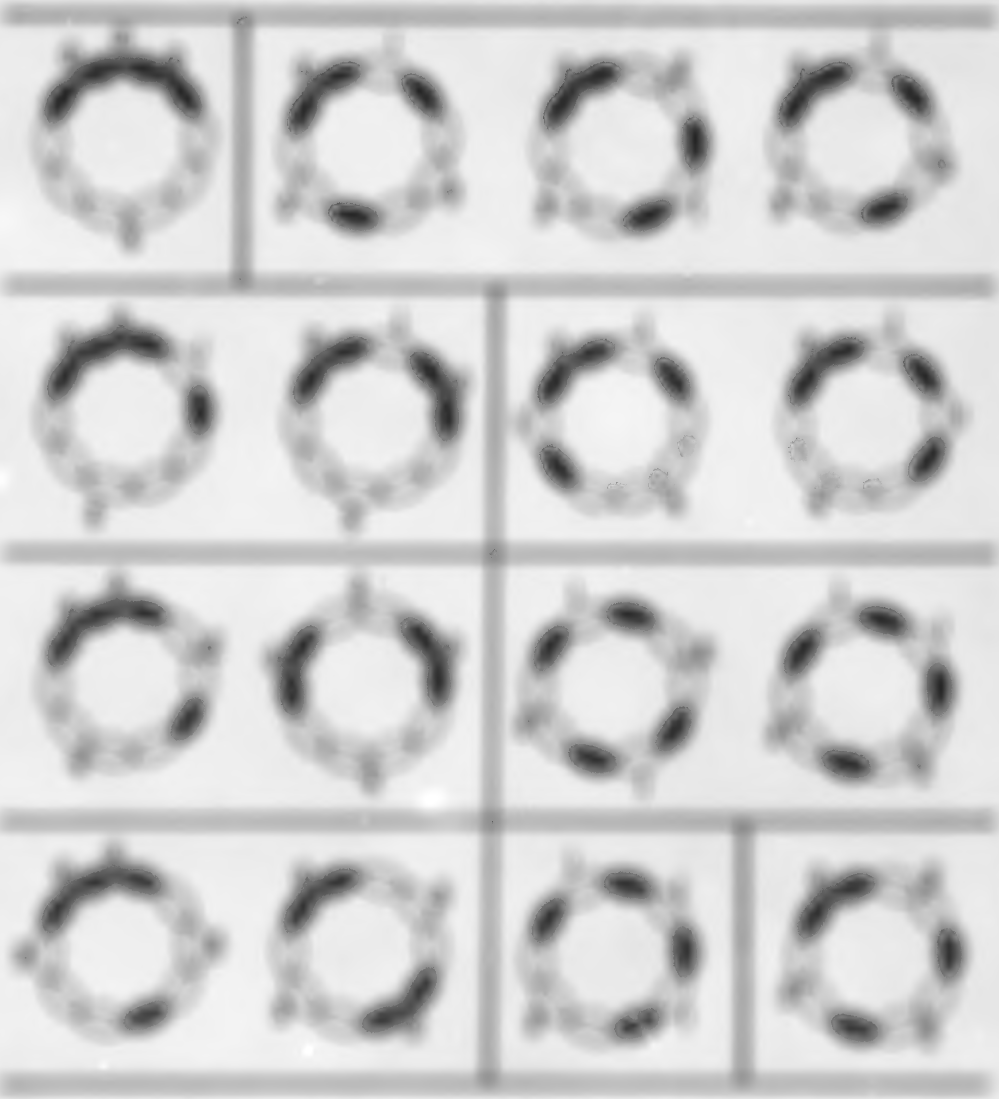


[illegible]

[illegible]

[illegible]

[illegible]



Record the number of arrangements of four red and six white beads and compare this number (16) with the estimates recorded earlier in the lesson.

NECKLACE

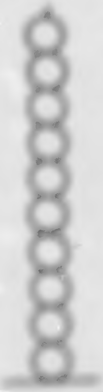
3 red - 7white	8 ways
4 red - 6 white	16 ways

Exercise 2

T: Do you remember what they do with the beads in the village where Theophilus' friend lives?

S: They put the beads on poles in front of young peoples' houses.

Draw a picture of a pole with beads on it.



T: Do you remember how many ways there are to arrange three red and seven white beads on a pole?

S: One hundred twenty.

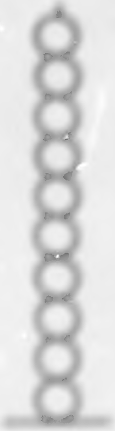
Make a chart similar to the one in Exercise 1.

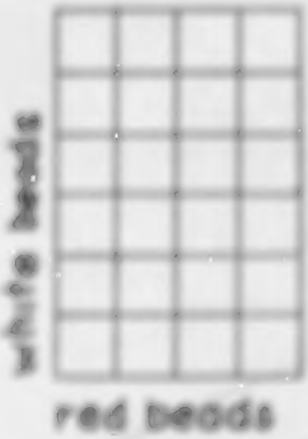
POLE

3 red - 7white	120 ways
4 red - 6 white	

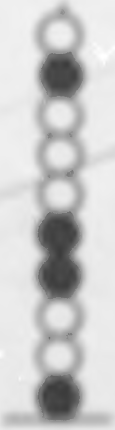
T: Now we want to find the number of ways there are to arrange four red and six white beads on a pole. How can we solve this problem?

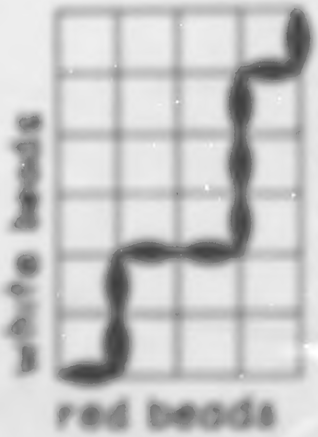
If no one suggests making a grid consisting of four rows and six columns, ask how the problem was solved in the last lesson on this topic (Exercise 2 of Lesson P2.1).





Invite one student to color the beads on the pole and then another to indicate how that arrangement is shown on the grid.





T: So in this setting we are asking how many paths there are from here (point to the lower left corner) to here (point to the upper right corner).

Hold up a copy of the Worksheet [illegible] and ask the students to find the total number of arrangements. You may need to help students individually to fill in the numbers at the grid points.* (This method is explained in Lesson [illegible] and in THE HIDDEN TREASURE.)

T: How many arrangements are there?

S: Two hundred ten.

Record this in the chart for the pole.

POLE

3 red - 7 white	120 ways
4 red - 6 white	210 ways

Compare these results with the corresponding results for the necklace.

NECKLACE

3 red - 7 white	8 ways
4 red - 6 white	[illegible] ways

POLE

3 red - 7 white	120 ways
4 red - 6 white	210 ways

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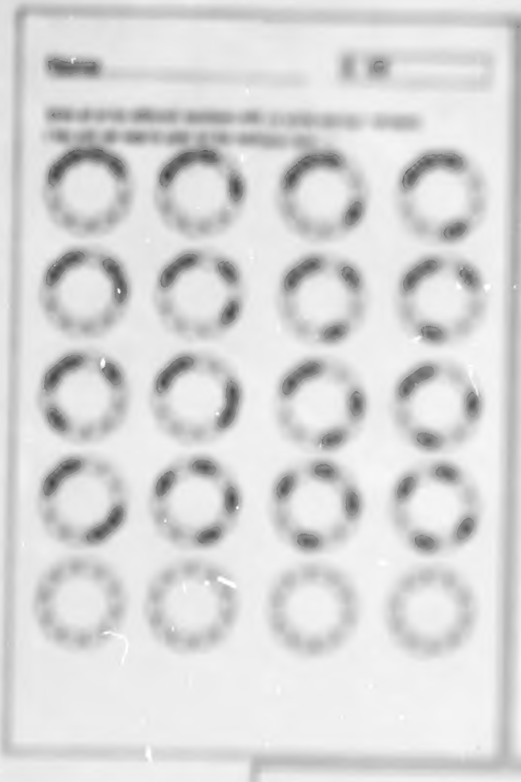
[*] See the answer key to Worksheet [illegible] on the next page.



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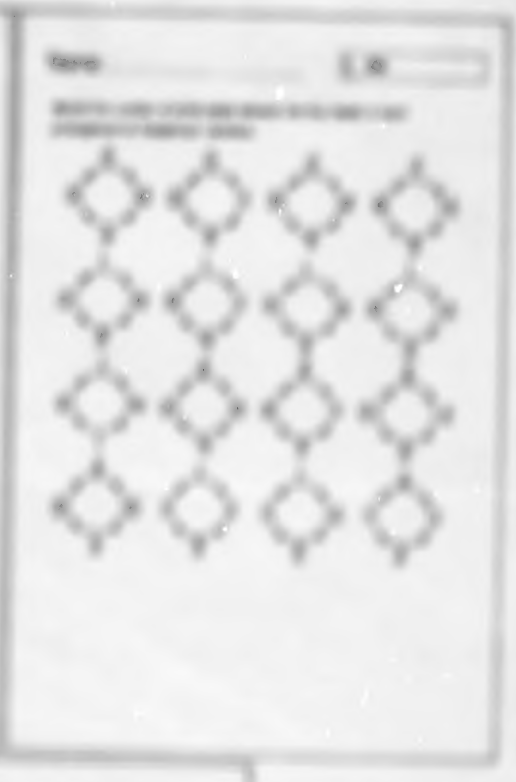
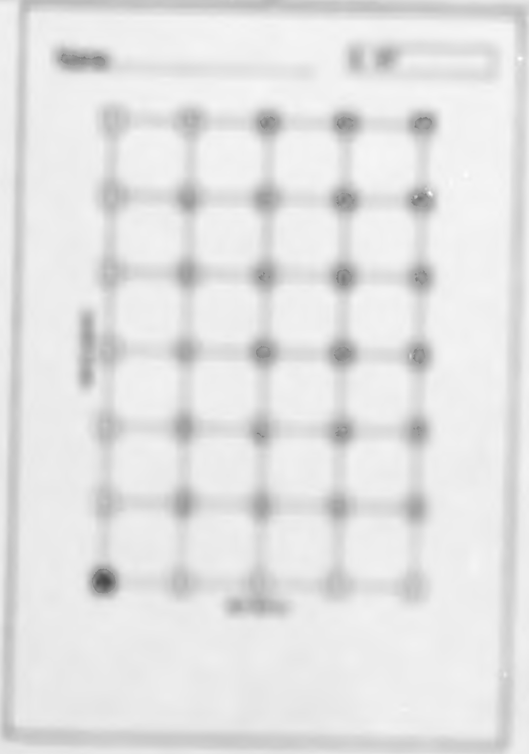


Name



Name

3

THE TABLE GAME

In the CSMP curriculum prior to Part V of the Intermediate Grades Program (IG-V), we have seen the String Game develop from a simple two-string game involving A-Blocks with color, size, and shape as attributes to a three-string game involving numbers with attributes such as multiples of 4, positive prime numbers, and positive divisors of 27. Intuitive playing develops into strategic thinking through exercises carefully analyzing the kinds of information given through the plays and the starting clues. This game continues in Part V to provide an endless source of interesting situations.

Another game is introduced in IG-V, the Table Game. The emphasis in the Table Game is on operations — on finding the images of ordered pairs of numbers under an operation, whereas in the String Game the emphasis is on classification — on building sets of numbers (or A-Blocks). But both games serve a common purpose, to develop strategic thinking. The Table Game leaps into the analysis level quickly, perhaps overly, because of the extensive analysis already done with the String Game. The correct placement of a number (or A-Block) immediately limits the possibilities for the secret string or strings in the String Game or for the secret operation in the Table Game. However, in the Table Game the incorrect placement of a number also immediately limits the possibilities for the operation, whereas in the String Game this is generally not the case.

The Table Game, as its name suggests, involves the table for an operation. The secret operation is one of twenty-two possibilities, and entries made during the game gradually reveal its identity. The rules of the game encourage careful analysis of entries made early in the game and sound knowledge of the operation and correct reading of an operation table later in the game.

In IG-V the Table Game itself is played only in the lessons described in Section 3.4 after those lessons leading up to it. In IG-VI (next semester) the game continues to be played and is integrated with other games involving strategic thinking.

CAPSULE LESSON SUMMARIES

Section 5.1: Introduce table operations through a Guess My Rule activity. Complete and discuss the patterns in a truth table for the relation "is a divisor of" and in a truth table for the relation [illegible]. Review the operation [illegible] and introduce the operation [illegible]. Given a list of six possible operations, determine the operation for a 3 by 3 table by making trial entries until the operation is evident.
(Worksheet [illegible])

Section 5.2: Extend the activities of Section 5.1 to include more operations, with an emphasis on:

[illegible]

(Worksheet [illegible])

Section 5.3: Review and complete a table for the operation ⊓ (greatest common divisor); review and complete a truth table for the relation [illegible]. Provided with a list of fifteen possible operations, determine the operation for:

- a 3 by 3 table by analyzing the given entries;
- a 4 by 4 table by making trial entries until the operation is evident.

(Worksheets [illegible])

Section 5.4: Introduce the truth table for the relation "is a multiple of" through a Guess My Rule activity and review the operation ⊔ (least common multiple). Provide a list of twenty-two operations and ask which of these operations have tables with various characteristics. Present the Table Game. Teams take turns making trial entries in a 4 by 4 table until someone correctly identifies which operation the table is for.

L6 ..... [illegible]

CAPSULE LESSON SUMMARY

[illegible]

MATERIALS

Teacher: None

Student: Paper; pencil; Worksheet L6

DESCRIPTION OF LESSON

Exercise 1

Students should have paper and pencils for this exercise.

T: See if you can guess my rule.

NOTE: The rule assigns 1 to an (ordered) pair of numbers if the first number is a divisor of the second and assigns 0 if it is not. Do not tell the rule to the students at this time.

Write these number stories on the board.

$$2 \ast 4 = 1$$
$$3 \ast 9 = 1$$
$$3 \ast 8 = 0$$
$$6 \ast 2 = 0$$
$$2 \ast 6 = \square$$

[illegible]

[illegible]

Walk around the room and ask students whether or not they have found the correct answer. Then let someone answer aloud.

S: [illegible]

Record "1" in the box.

$$2 \bullet 6 = \boxed{1}$$

In the same manner ask students to complete the following number puzzles. [illegible] one at a time. Answers are in the boxes. When many students have found the answer to a problem, let one of them record it at the front of the class.

$$5 \bullet 5 = \boxed{1}$$

$$4 \bullet 7 = \boxed{0}$$

T: What is my rule?

S: The result is 1 if the first number is a divisor of the second; otherwise, it is 0.

[illegible]

[illegible]

[illegible]

E. '[illegible] is a divisor of [illegible]' is true, so [illegible] ∘ [illegible] = 1.
'[illegible] is a divisor of [illegible]' is true, so [illegible] ∘ [illegible] = 1.
⋮ ⋮
'[illegible] is a divisor of [illegible]' is false, so [illegible] ∘ [illegible] = 0.
Since we only consider whether a sentence about numbers is true or false, this rule has to do with a truth table; ∘ could stand for the [illegible] [illegible], 'is a divisor of'.

Begin a list of operations on one side of the board.

T_9

Repeat this activity using the following rule:

NOTE: This rule assigns 1 to a pair of numbers if the numbers are equal modulo [illegible] and assigns 0 if they are not. Do not tell the rule to the students.

Write the following number phrases on the board. The first four number phrases are the given information from which the students discover the rule; the last two the students complete one at a time as a way of checking whether or not they have found the correct rule. Answers are in the boxes.

5 ∘ 7 = 1
[illegible] ∘ [illegible] = 0
[illegible] ∘ [illegible] = 0
[illegible] ∘ [illegible] = 1
[illegible] ∘ [illegible] = $\boxed{0}$
[illegible] ∘ [illegible] = $\boxed{1}$

84

L6

T: [illegible]

S: [illegible]

S: [illegible]

[illegible]

T: [illegible]

[illegible]

$$\overline{T}_2$$

$$\overline{T}_{\equiv_2}$$

[illegible]

T_2	1	2	3	4	5	6	7	8	9
[illegible]									
[illegible]									
[illegible]									
[illegible]									
[illegible]									
[illegible]									
[illegible]									
[illegible]									
[illegible]									

[illegible]

[illegible]

[illegible]

[illegible] Why?

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

1.6

[illegible]

a. [illegible]

b. [illegible]

[illegible]

[illegible]

c. [illegible]

[illegible]

[illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible]

[illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible] [illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

T: To multiply two numbers modulo 5, we can multiply the numbers in the usual way and then subtract fives until we get one of the numbers 0, 1, 2, 3, 4.

S: 3 × 6 = 18 and 3 is the remainder when you divide 18 by 5.

T: That's another way to multiply modulo 5. If we divide a number by 5, what are the possible remainders?

S: 0, 1, 2, 3, and 4.

As a collective activity complete these number stories with your students. Answers are in the boxes.

$$6 \times_5 5 = \boxed{0}$$

$$8 \times_5 4 = \boxed{2}$$

$$3 \times_5 2 = \boxed{1}$$

$$9 \times_5 2 = \boxed{3}$$

$$3 \times_5 5 = \boxed{0}$$

$$7 \times_5 7 = \boxed{4}$$

$$7 \times_5 8 = \boxed{1}$$

Include $\times_5$ in the list of operations forming on the board and then erase the board except for the list and the table.

T: I'll also add to our list the usual operations + and ×.

L 6

You should have this table and this list of operations on the board.

*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

T_D

$T_{=2}$

$+_{10}$

$\times_5$

$+$

$\times$

Exercise 2

T: We're going to play a kind of guess my rule game on this table. I will choose one of these six operations (point to the list); you will try to figure out which one. You put any numbers you can choose in the table and I'll tell you whether or not the entries are correct.

Choose $+_{10}$ as your secret operation, but do not tell the students. Invite a student to put a number in the table. Suppose the student puts 6 in the table as in the next illustration.

*	1	2	3	4	5	6	7	8	9
1									
2				6					
3									
4									
5									
6									
7									
8									
9									

T_D

$T_{=2}$

$+_{10}$

$\times_5$

$+$

$\times$

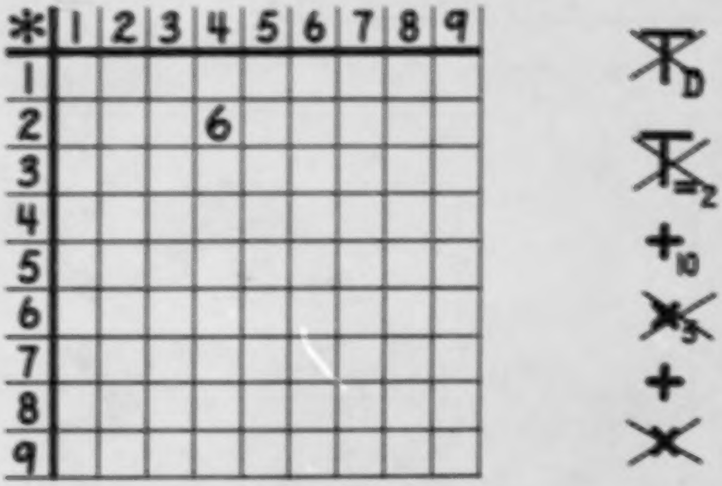
T: $2 * 4 = 6$. That's correct! Are there any of these operations (point to the list) that we can cross out?

S: Cross out "T_D" and "$T_{=2}$", because there are only zeros and ones in those tables.

S: Cross out "$\times$" because $2 \times 4 = 8$.

S: Cross out "$\times_5$" because $2 \times_5 4 = 3$.

S: We also know that the operation cannot be $\times_5$ because 6 is not one of the numbers 0, 1, 2, 3, 4.



*	1	2	3	4	5	6	7	8	9
1									
2				6					
3									
4									
5									
6									
7									
8									
9									

~~T_D~~
~~$T_{=2}$~~
$+_{10}$
~~$\times_5$~~
$+$
~~$\times$~~

T: We have only two possibilities left, $+_{10}$ and $+$. Who would like to put another number in the table?

L 6

Suppose a student puts 14 in the table as in the next illustration.

*	1	2	3	4	5	6	7	8	9
1									
2				6					
3									
4									
5									
6								14	
7									
8									
9									

$\cancel{T_D}$

$\cancel{T_{\geq 2}}$

$+_{10}$

$\cancel{\times_5}$

$+$

$\cancel{\times}$

T: 6 * 8 does not equal 14.

*	1	2	3	4	5	6	7	8	9
1									
2				6					
3									
4									
5									
6								~~14~~	
7									
8									
9									

$\cancel{T_D}$

$\cancel{T_{\geq 2}}$

$+_{10}$

$\cancel{\times_5}$

$+$

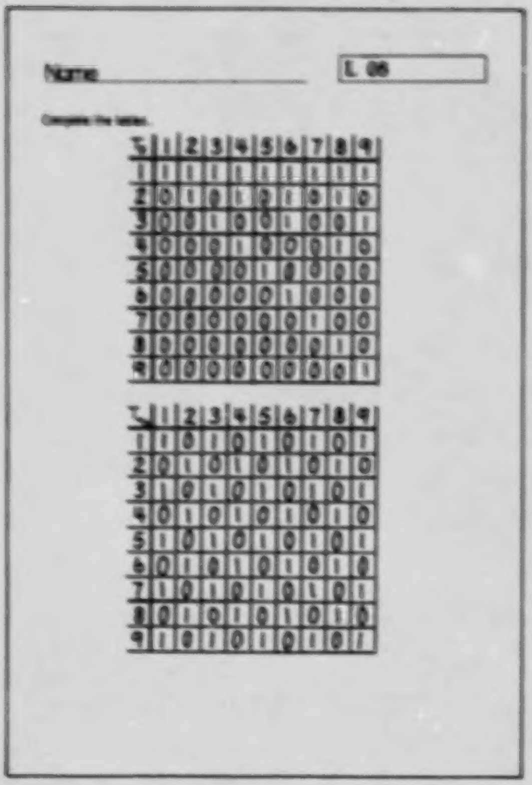
$\cancel{\times}$

T: With this new information, do we know which of these (point to "$+_{10}$" and "+") is my secret operation?

S: It must be $+_{10}$, because 6 + 8 does equal 14 and "14" is crossed out.

On the list cross out "+" and circle "$+_{10}$".

NOTE: Depending on your students' choices, this activity could require two or more than two entries in the table to determine the operation.



Name____________________ L 08

Complete the tables.

↘	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0
3	0	0	1	0	0	1	0	0	1
4	0	0	0	1	0	0	0	1	0
5	0	0	0	0	1	0	0	0	0
6	0	0	0	0	0	1	0	0	0
7	0	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	0	1

↘	1	2	3	4	5	6	7	8	9
1	1	0	1	0	1	0	1	0	1
2	0	1	0	1	0	1	0	1	0
3	1	0	1	0	1	0	1	0	1
4	0	1	0	1	0	1	0	1	0
5	1	0	1	0	1	0	1	0	1
6	0	1	0	1	0	1	0	1	0
7	1	0	1	0	1	0	1	0	1
8	0	1	0	1	0	1	0	1	0
9	1	0	1	0	1	0	1	0	1

L8

3.2 Table Game #2

CAPSULE LESSON SUMMARY

Extend the activities of Lesson L3.1 to include more operations, with an emphasis on:

$$\times_5,\ -_5,\ +_5,\ \times_3,\ -_3,\ +_3$$

MATERIALS

Teacher: None
Student: Paper; pencil; Worksheet L09

DESCRIPTION OF LESSON

Exercise 1

T: See if you can guess my rule.

NOTE: The rule assigns to a pair of numbers the larger of the two numbers or the number if they are the same (for example, it assigns 7 to the pair (7, 7)). Do not tell the rule to the students at this time.

Write these number stories on the board.

$$3 * 5 = 5$$
$$9 * 4 = 9$$
$$4 * 9 = 9$$
$$6 * 6 = 6$$
$$1 * 8 = \square$$

T (pointing to the last number story): If you know my rule, what number is 1 * 8?

S: 8.

Record "8" in the box.

T: What is my rule?

S: The result is the larger of the two numbers if the numbers are not the same. If the numbers are the same, the result is that number.

T: We'll use this symbol . . .

$$\uparrow$$

to indicate this rule.

To reinforce the notation, ask the students to complete these number stories. Answers are in the boxes.

$$2 \uparrow 5 = \boxed{5}$$
$$3 \uparrow 3 = \boxed{3}$$
$$0 \uparrow 1 = \boxed{1}$$

T: What do you think an arrow pointing down means?

S: The result is the smaller of the two numbers.

T: . . . and if the numbers are the same?

S: The result is that number.

Include ↓ in a list of operations forming on the board.

↑

↓

Erase the answers to the last three number stories on the board and change the arrows in them so that they point downward. Ask the students to complete the resulting number stories. Answers are in the boxes.

$$2 \downarrow 5 = \boxed{2}$$
$$3 \downarrow 3 = \boxed{3}$$
$$0 \downarrow 1 = \boxed{0}$$

Repeat this activity using a new rule.

NOTE: This new rule assigns 1 to a pair of numbers if the first number is smaller than the second and assigns 0 if it is not. Do not tell the rule to the students.

Write the following number stories on the board. The first four number stories are the given information from which the students discover the rule; the last three the students complete one at a time as a way of checking whether or not they have found the correct rule. Answers are in the boxes.

$$3 * 7 = 1$$
$$1 * 2 = 1$$
$$8 * 5 = 0$$
$$5 * 8 = 1$$
$$2 * 4 = \boxed{1}$$
$$9 * 6 = \boxed{0}$$
$$5 * 5 = \boxed{0}$$

T: What is my rule?

S: The result is 1 if the first number is smaller than the second; otherwise, the result is 0.

Point to the appropriate number stories as you say:

T: "3 is smaller than 7" is true, so 3 * 7 = 1.
"8 is smaller than 5" is false, so 8 * 5 = 0.
Since we only consider whether a sentence about numbers is true or false, we say this rule has to do with a truth table — a truth table for <.

Include $T_<$ in the list of operations forming on the board.

$$\uparrow$$
$$\downarrow$$
$$T_<$$

L8

T: There also is a truth table for >. It is the table for a rule in which the result is 1 if the first number is larger than the second and is 0 otherwise. For example, 8 * 2 would be 1 and 4 * 9 would be 0.

Include $T_>$ in the list of operations.

$$\uparrow$$

$$\downarrow$$

$$T_<$$

$$T_>$$

T: Do you remember the operation $\times_5$ we talked about last week? What number is $6 \times_5 6$?

S: 1, because 6 x 6 = 36 and we subtract fives until we get one of the numbers 0, 1, 2, 3, or 4.

S: 1, because 6 x 6 = 36 and we get a remainder of 1 when we divide 36 by 5.

If the first of these two explanations is not given, suggest it yourself as one way of looking at the operation. Write this list of numbers on the board.

0 1 2 3 4

T: What number is 9 - 3?

S: 6.

T: What number do you think $9 -_5 3$ (read: nine minus three modulo five) is?

S: 1.

T: 1 is correct. How did you get 1?

S: I did two subtractions: $9 - 3 = 6$ and $6 - 5 = 1$.

T: 6 is larger than any of the numbers 0, 1, 2, 3, or 4; so we subtract fives until we get one of them. In this case, we need to subtract five just once.
What number is $7 -_5 4$?

S: 3.

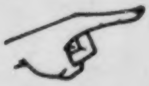
T: What number is $2 - 8$?

S: $^-6$.

T: What number is $2 -_5 8$? Remember the answer must be one of the numbers 0, 1, 2, 3, or 4.

If some students suggest that the answer is 1, they very likely calculated "$8 -_5 2$" instead of "$2 -_5 8$". If no one knows the correct answer (4), give it yourself.

T: When the usual difference is less than any of these numbers . . .



0 1 2 3 4

we add fives until we get one of them. $2 - 8 = {}^-6$, $^-6 + 5 = {}^-1$, and $^-1 + 5 = 4$ (point to "4" in the list).

L 8

Go over a few more examples to help the students learn how to use this operation when negative numbers arise. For example,

$$1 -_5 3 = \boxed{3}$$

$$6 -_5 9 = \boxed{2}$$

List the operations $\times_5$, $-_5$, and $+_5$ on the board near the list of five numbers.

$\times_5$

$-_5$ $\qquad$ 0 1 2 3 4

$+_5$

T (pointing to "$+_5$"): Addition modulo 5 is done similarly. For example, $6 + 7 = 13$, so $6 +_5 7 = 3$.

Point to the symbols "$\times_5$", "$-_5$", and "$+_5$".

T: To do any of these operations, we do the usual operation and then we add or subtract fives to the resulting number until we get one of the numbers: 0, 1, 2, 3, or 4.

Write symbols for modulo 3 operations under the modulo 5 operation symbols.

$$\begin{array}{ll} \times_5 & \\ -_5 & 0 \quad 1 \quad 2 \quad 3 \quad 4 \\ +_5 & \end{array}$$

$$\begin{array}{l} \times_3 \\ -_3 \\ +_3 \end{array}$$

T: If we multiply modulo 3, subtract modulo 3, or add modulo 3, what numbers do you think we can get?

S: 0, 1, and 2.

If no one suggests these numbers, suggest them yourself. List them next to the modulo 3 operation symbols.

$$\begin{array}{ll} \times_5 & \\ -_5 & 0 \quad 1 \quad 2 \quad 3 \quad 4 \\ +_5 & \end{array}$$

$$\begin{array}{ll} \times_3 & \\ -_3 & 0 \quad 1 \quad 2 \\ +_3 & \end{array}$$

L 8

T: What number is 4 + 6? (10)

What number is $4 +_3 6$?

S: 1.

T: How did you get 1?

S: I started with 10 and subtracted 3 as many times as necessary until I got either 0, 1, or 2.

In the same way go over these problems. Answers are in the boxes.

$$5 \times_3 4 = \boxed{2}$$

$$7 -_3 9 = \boxed{1}$$

Emphasize that 7 - 9 = ⁻2 so we add as many threes as necessary starting at ⁻2 until we get one of the numbers 0, 1, or 2.

Distribute copies of Worksheet L09. Students should fill in the tables as an individual activity. As they are working, draw a table on the board and make additions to the list of operation symbols as indicated in the next illustration. The rest of the board can be erased.

*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

$\times_5$
$-_5$
$+_5$
$\times_3$
$-_3$
$+_3$

$\times$
$-$
$+$
$\uparrow$
$\downarrow$
$T_<$
$T_>$
$T_{=_2}$

After many students have completed both tables, discuss patterns in the tables. A sample dialogue is given.

T: Look at your table (on the worksheet) for $-_3$. What patterns do you see?*

S: The same number is never next to itself.

S: There is always a "0" before a "2", a "2" before a "1", and a "1" before a "0" (horizontal pattern).

S: Along a diagonal the entries are all the same number.

T: Which rows are the same?

S: The first, the fourth, and the seventh are the same.

S: The second, the fifth, and the eighth are the same.

S: The third, the sixth, and the ninth are the same.

T: Look at your table (on the worksheet) for $\times_3$. Which rows have all "0"'s in them?

S: The third, the sixth, and the ninth.

T: Do you know why?

S: Because the numbers 3, 6, and 9 are multiples of 3.

---

* Refer to the answer key on page 104.

If your students would benefit from completing the table for $\times_3$ collectively, fill in the table on the board. Afterward, erase the entries.

Keep the table and the list on the board for Exercise 2.

Exercise 2

T: I am thinking of one of these operations (point to the list). See if you can figure out which one. You can choose numbers to put into the table. I'll tell you whether or not the entries are correct.

Choose ↓ as the operation, but do not tell the students. Invite students one at a time to put numbers in the table. After each entry, discuss which operations could not be the secret operation and cross them off the list. Since there are many operations to consider, varying the order in which they are considered versus the order in which they are listed could make the analysis more interesting and move faster. For example, suppose a student puts 1 in the table as in the next illustration and is told that the entry is correct.

*	1	2	3	4	5	6	7	8	9
1							1		
2									
3									
4									
5									
6									
7									
8									
9									

$\times_5$
$-_5$
$+_5$
$\times_3$
$-_3$
$+_3$

$\times$
$-$
$+$
$\uparrow$
$\downarrow$
$T_<$
$T_>$
$T_{=2}$

This is one possible order for considering the operations. If your students spontaneously begin ruling out operations, do not direct them otherwise.

- Consider first the operations ↑ and ↓.

  T: Is 1 the larger of the pair 1 and 7? (No)
  Is 1 the smaller? (Yes)

  Cross off "↑".

- Consider all the addition operations.

  T: Is $1 + 7 = 1$? (No)
  Is $1 +_5 7 = 1$? (No)
  Is $1 +_3 7 = 1$? (No)

  Cross off all the addition operations.

- Consider next $T_<$ and $T_>$.

  T: Is $1 < 7$? (Yes)
  Is $1 > 7$? (No)

  Cross off "$T_>$".

- Consider all the multiplication operations.

  T: Is $1 \times 7 = 1$? (No)
  Is $1 \times_5 7 = 1$? (No)
  Is $1 \times_3 7 = 1$? (Yes)

  Cross off "$\times$" and "$\times_5$".

- Consider $T_{=_2}$.

  T: Is $1 =_2 7$? That is, are 1 and 7 both even or both odd?
  (Yes, both odd)

- Consider all the subtraction operations.

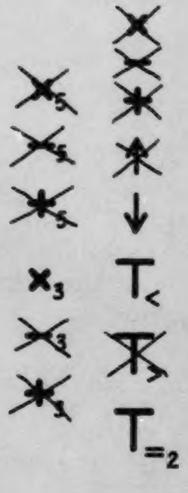
  T: Is $1 - 7 = 1$? (No)
  Is $1 -_5 7 = 1$? (No)
  Is $1 -_3 7 = 1$? (No)

  Cross off all the subtraction operations.

L 8

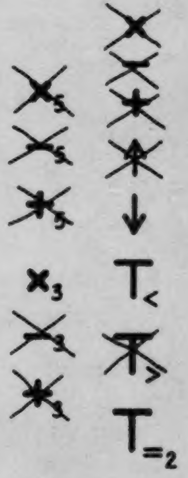
All but four operations should be crossed off.

*	1	2	3	4	5	6	7	8	9
1							1		
2									
3									
4									
5									
6									
7									
8									
9									


$\times_5$
$+_5$
$*_5$
$\times_3$
$+_3$
$*_3$
$\times$
$+$
$*$
$*$
↓
$T_<$
$T_>$
$T_{=_2}$

Suppose the second entry in the table is incorrect.

*	1	2	3	4	5	6	7	8	9
1							1		
2									
3									
4						~~1~~			
5									
6									
7									
8									
9									


$\times_5$
$+_5$
$*_5$
$\times_3$
$+_3$
$*_3$
$\times$
$+$
$*$
$*$
↓
$T_<$
$T_>$
$T_{=_2}$

S: The operation is not $T_{=_2}$ because "$4 =_2 6$" is true (4 and 6 are both even), so $4\ T_{=_2}\ 6 = 1$.

S: The operation is not $T_<$ because 4 is less than 6, so $4\ T_<\ 6 = 1$.

Cross off $T_{=2}$ and $T_{<}$.

*	1	2	3	4	5	6	7	8	9
1							1		
2									
3									
4						~~1~~			
5									
6									
7									
8									
9									

$\cancel{\times_5}$
$\cancel{-_5}$
$\cancel{*_5}$
$\times_3$
$\cancel{-_3}$
$\cancel{*_3}$

$\cancel{\times}$
$\cancel{-}$
$\cancel{*}$
$\cancel{\uparrow}$
$\downarrow$
$\cancel{T_{<}}$
$\cancel{T_{>}}$
$\cancel{T_{=2}}$

Continue in this way analyzing entries made by students until the operation is determined. In the game described, a good choice of third entry would distinguish whether the operation is $\times_3$ or ↓ .

If there is time remaining, erase the entries in the table and recopy the list of operations on the board. Repeat the activity for other choices of the "secret" operation, for example, $\times_3$ or $T_{=2}$.

Name ______________________ L 09

Complete the tables.

$-_3$	1	2	3	4	5	6	7	8	9
1	0	2	1	0	2	1	0	2	1
2	1	0	2	1	0	2	1	0	2
3	2	1	0	2	1	0	2	1	0
4	0	2	1	0	2	1	0	2	1
5	1	0	2	1	0	2	1	0	2
6	2	1	0	2	1	0	2	1	0
7	0	2	1	0	2	1	0	2	1
8	1	0	2	1	0	2	1	0	2
9	2	1	0	2	1	0	2	1	0

$\times_3$	1	2	3	4	5	6	7	8	9
1	1	2	0	1	2	0	1	2	0
2	2	1	0	2	1	0	2	1	0
3	0	0	0	0	0	0	0	0	0
4	1	2	0	1	2	0	1	2	0
5	2	1	0	2	1	0	2	1	0
6	0	0	0	0	0	0	0	0	0
7	1	2	0	1	2	0	1	2	0
8	2	1	0	2	1	0	2	1	0
9	0	0	0	0	0	0	0	0	0

3.3 Table Game #3

CAPSULE LESSON SUMMARY

Review and complete a table for the operation $\sqcap$ (greatest common divisor); review and complete a truth table for the relation $=_4$. Provided with a list of fifteen possible operations, determine the operation for

- a 2 by 2 table by analyzing its given entries;
- a 9 by 9 table by making trial entries until the operation is evident.

MATERIALS

Teacher: Colored chalk
Student: Pencil; Worksheets L10, L11, L12

DESCRIPTION OF LESSON

Exercise 1

Write this list of operations to the side of the board.

$+$
$\times$
$-$
$+_3$
$\times_3$
$-_3$
$+_{10}$
$\times_{10}$
$-_{10}$

$\uparrow$
$\downarrow$
$\sqcap$
T_D
$T_{=2}$
$T_{=4}$

L 10

T: Today we will try to figure out which of these operations a table is for by looking at some of its entries. But first let's review some of these operations.

Write this number story on the board.

$$8 \sqcap 12 = \square$$

T: What number goes in the box?

S: 4.

T: How did you get 4?

S: I found the greatest common divisor of 8 and 12. 4 is a divisor of 8 and a divisor of 12 and 4 is the largest divisor of both.

Explain the operation yourself if necessary. Write these number stories on the board and complete them as a collective exercise. Answers are in the boxes.

$$14 \sqcap 49 = \boxed{7}$$

$$15 \sqcap 45 = \boxed{15}$$

$$56 \sqcap 64 = \boxed{8}$$

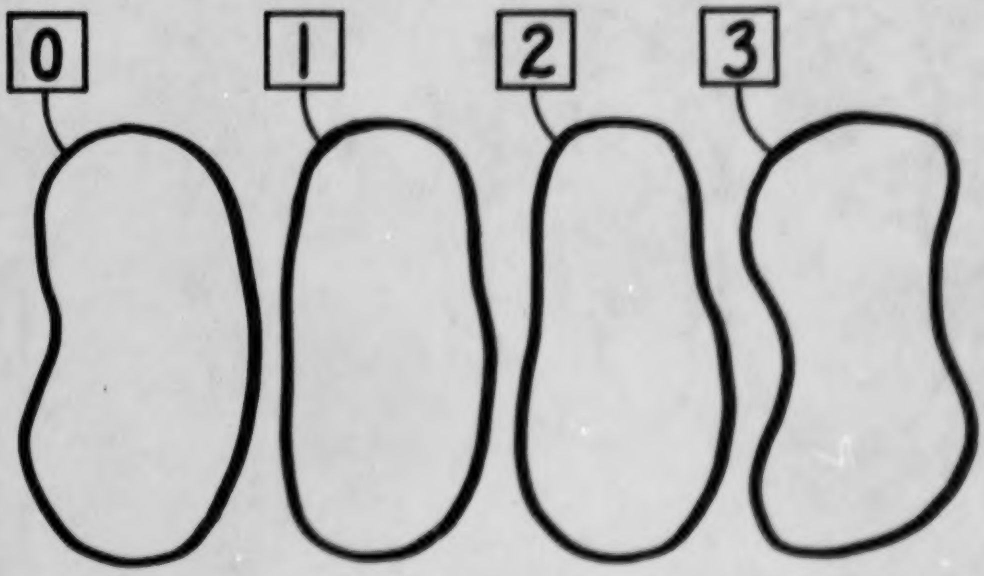
Erase the board and then write these number sentences on the board.

$$10 =_4 24$$

$$17 =_4 37$$

T: Do you remember the relation $=_4$ (read: equal modulo 4)

Let the students recall as much as they can about this relation. Draw four strings on the board.



S: The 0-string is for multiples of 4, the 1-string is for numbers which are 1 more than a multiple of 4, the 2-string is for numbers which are 2 more than a multiple of 4, and the 3-string is for numbers which are 3 more than a multiple of 4.

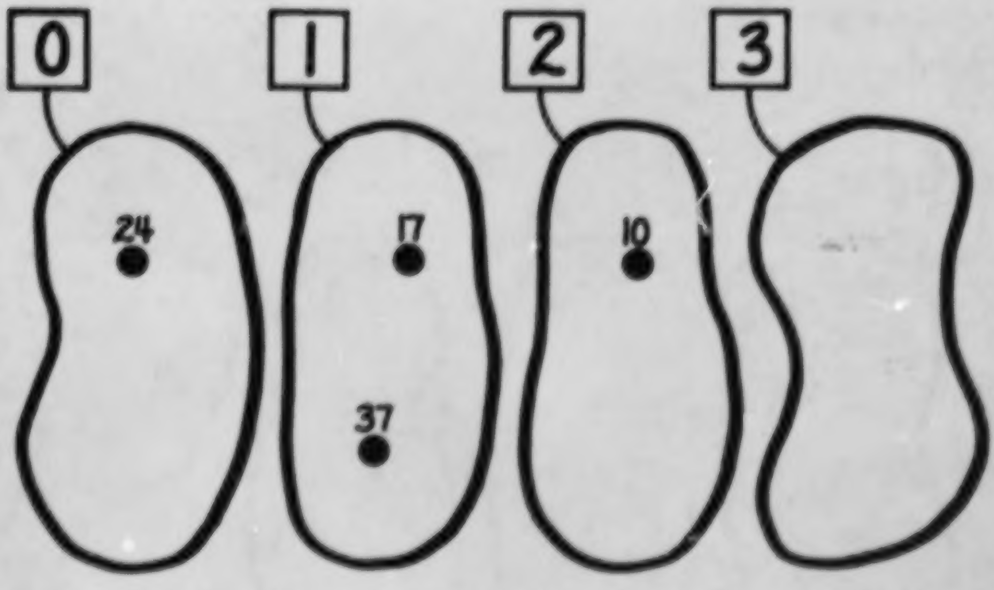
You may need to identify the strings yourself.

T: Are the number stories on the board true or false?

S: 10 is not equal to 24 modulo 4. 10 is in the 2-string and 24 is in the 0-string.

L 10

S: 17 is equal to 37 modulo 4, because 17 and 37 are both 1 more than a multiple of 4. They both belong in the 1-string.



T: So if the operation * is $T_{=_4}$, 10 * 24 = 0 because "10 $=_4$ 24" is false and 17 * 37 = 1 because "17 $=_4$ 37" is true.

Form a list on the board.

10 * 24 = 0	17 * 37 = 1

T: What are some other pairs of numbers which are equal modulo 4? . . . which are not equal modulo 4?

Continue until several pairs of each type have been suggested.

10 * 24 = 0
71 * 72 = 0
20 * 30 = 0
51 * 90 = 0
2 * 7 = 0

17 * 37 = 1
106 * 46 = 1
25 * 13 = 1
1 * 5 = 1
28 * 36 = 1

T: Given two numbers, is there a simple way to tell whether the result will be 1 or 0 with this operation?

S: Subtract the two numbers. If you get a multiple of 4, the result is 1; otherwise, the result is 0.[#]

Reinforce this idea by looking at several examples in the list as in the following dialogue.

T: 37 - 17 = . . . ?

S: 20.

T: Is 20 a multiple of 4?

S: Yes.

T: So 17 * 37 = 1.
24 - 10 = . . . ?

[#] The order in which the two numbers are subtracted is of no importance here; you either get a multiple of 4 in both cases or in neither. For example,

37 - 17 = 20 and 20 is a multiple of 4;
17 - 37 = $^{-}20$ and $^{-}20$ is a multiple of 4.

S: 14.

T: Is 14 a multiple of 4?

S: No.

T: So 10 * 24 = 0.

Distribute copies of Worksheet L10. Ask the students to complete the tables. As they are working, draw a table for ⊓ on the board. After most of the students have completed the tables, invite some students each to fill in a row of the table on the board. Continue until the table is complete.

⊓	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1
3	1	1	3	1	1	3	1	1	3
4	1	2	1	4	1	2	1	4	1
5	1	1	1	1	5	1	1	1	1
6	1	2	3	2	1	6	1	2	3
7	1	1	1	1	1	1	7	1	1
8	1	2	1	4	1	2	1	8	1
9	1	1	3	1	1	3	1	1	9

T: Which numbers occur in the row for 5?

S: 1 and 5.

T: Which numbers occur in the row for 6?

S: 1, 2, 3, and 6.

Encourage students to look for patterns in this table.

T: Look at your table (on the worksheet) for $T_{=4}$. Do you notice any patterns?[#]

S: In each row, there are three "0"'s between "1"'s. The same for each column.

S: The diagonal has all 1's.

Erase the operation symbol and the entries in the table. The table will be used in Exercise 3.

Exercise 2

On one side of the list of operation symbols, draw this completed 2 by 2 table.

$+$
$\times$
$-$
$+_3$
$\times_3$
$-_3$
$+_{10}$
$\times_{10}$
$-_{10}$

$\uparrow$
$\downarrow$
$\sqcap$
T_D
$T_{=2}$
$T_{=4}$

*	5	9
3	5	7
4	0	6

NOTE: The table is for the operation $\times_{10}$. Do not tell this to the class.

T: This table is for one of these operations (point to the list). Do you know which it is?

[#] See the answer key to Worksheet L10 on page 115.

L 10

For each operation a student suggests, check the four entries in the table. It then will become evident whether or not the table is for the operation named. Continue until the class discovers that the table is for $\times_{10}$.

Repeat this activity using this table. The operation is $+_3$.

$+$
$\times$
$-$
$+_3$
$\times_3$
$-_3$
$+_{10}$
$\times_{10}$
$-_{10}$

$\uparrow$
$\downarrow$
$\sqcap$
T_D
$T_{=2}$
$T_{=4}$

*	3	5
2	2	1
4	1	0

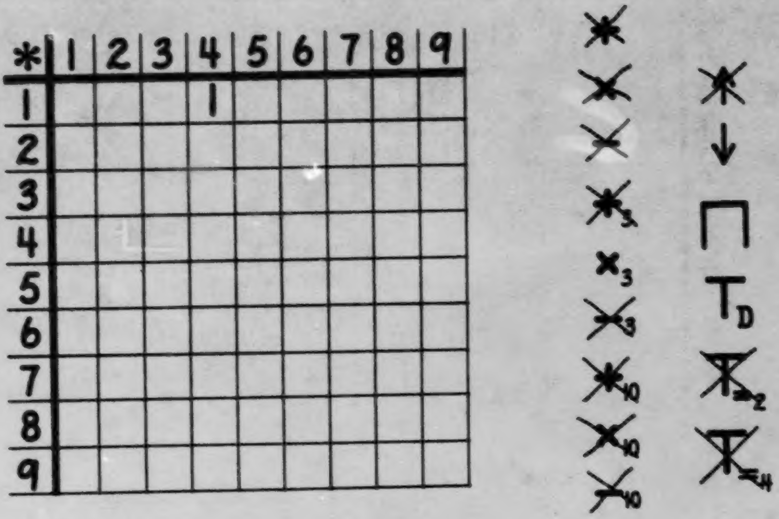
Erase only the 2 by 2 table. You will need the list of operation symbols and the 9 by 9 table for Exercise 3.

Exercise 3

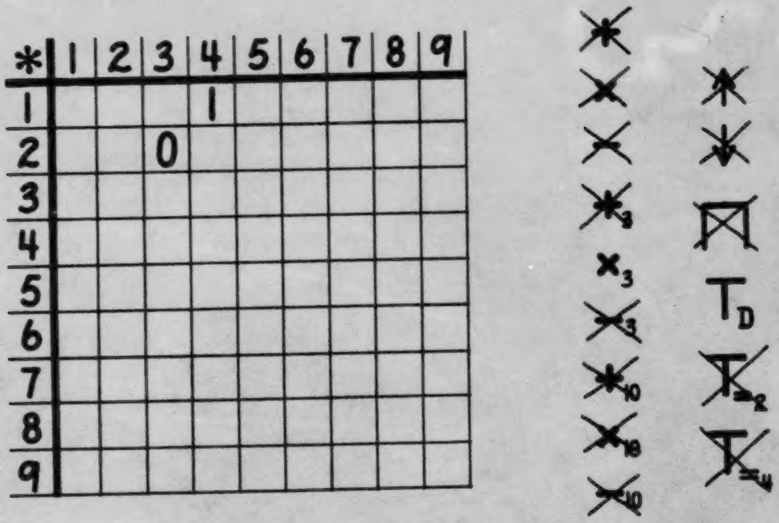
T: I am thinking of one of these operations (point to the list). See if you can figure out which one. You can choose numbers and put them into the table. I'll tell you whether or not the entries are correct.

Choose T_D as the "secret" operation, but do not tell the students. Invite students, one at a time, to put numbers in table. Indicate that an entry is correct, or that it is incorrect putting an "×" through it. After each entry, discuss which operations can be crossed off the list because the table could not be for any of them. Continue until the operation is determined.

A possible game is described in the sequence of pictures below and on the next page.



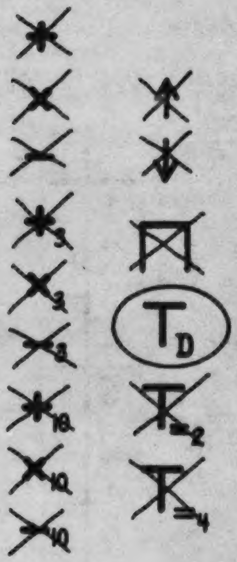
*	1	2	3	4	5	6	7	8	9
1				1					
2									
3									
4									
5									
6									
7									
8									
9									



*	1	2	3	4	5	6	7	8	9
1				1					
2			0						
3									
4									
5									
6									
7									
8									
9									

L 10

T_D	1	2	3	4	5	6	7	8	9
1				1					
2			0						
3						$\xcancel{8}$			
4									
5									
6									
7									
8									
9									


3
3
3
10
10
10
T_D
=2
=4

Worksheets L11 and L12 are available for independent work.

Name ____________________ **L 10**

Complete the tables.

⊓	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1
3	1	1	3	1	1	3	1	1	3
4	1	2	1	4	1	2	1	4	1
5	1	1	1	1	5	1	1	1	1
6	1	2	3	2	1	6	1	2	3
7	1	1	1	1	1	1	7	1	1
8	1	2	1	4	1	2	1	8	1
9	1	1	3	1	1	3	1	1	9

$T_{\equiv 4}$	1	2	3	4	5	6	7	8	9
1	1	0	0	0	1	0	0	0	1
2	0	1	0	0	0	1	0	0	0
3	0	0	1	0	0	0	1	0	0
4	0	0	0	1	0	0	0	1	0
5	1	0	0	0	1	0	0	0	1
6	0	1	0	0	0	1	0	0	0
7	0	0	1	0	0	0	1	0	0
8	0	0	0	1	0	0	0	1	0
9	1	0	0	0	1	0	0	0	1

Name ____________________ **L 11** *

Each table is for one of these.

$+$	$+_3$	$+_{10}$
$-$	$-_3$	$-_{10}$
$\times$	$\times_3$	$\times_{10}$
↑	⊓	$T_{\equiv 2}$
↓	T_D	$T_{\equiv 4}$

Label the tables.

↑	7	8
5	7	8
6	7	8

T_D	1	6
1	1	1
3	0	1

$-$	3	4
3	0	$^-1$
2	$^-1$	$^-2$

$T_{\equiv 2}$	6	9
3	0	1
7	0	1

Name ____________________ **L 12** **

Each table is for one of these.

$+$	$+_3$	$+_{10}$
$-$	$-_3$	$-_{10}$
$\times$	$\times_3$	$\times_{10}$
↑	⊓	$T_{\equiv 2}$
↓	T_D	$T_{\equiv 4}$

Using the information from the entries given, decide what each table is for and label it.

$-_3$	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6						0			
7			1						
8							1		
9									

$\times_{10}$	1	2	3	4	5	6	7	8	9
1									
2						2			
3								4	
4									
5		0							
6									
7									
8									
9									

L 12

3.4 Table Game #4

CAPSULE LESSON SUMMARY

Introduce the truth table for the "is a multiple of" relation through a Guess My Rule activity and review the operation $\sqcup$ (least common multiple). Provide a list of twenty-two operations and ask which of those operations have tables with various characteristics. Present the Table Game. Teams take turns making trial entries in a 9 by 9 table until someone correctly identifies which operation the table is for.

MATERIALS

Teacher: The Table Game Poster
Student: Table Game Analysis Pad; pencil

DESCRIPTION OF LESSON

Exercise 1

T: See if you can guess my rule.

NOTE: The rule assigns 1 to an ordered pair of numbers if the first number is a multiple of the second and assigns 0 if it is not. Do not tell the rule to the students at this time.

Write these number stories on the board.

$$4 * 8 = 0$$

$$9 * 3 = 1$$

$$3 * 9 = 0$$

$$4 * 5 = 0$$

$$5 * 4 = 0$$

$$12 * 6 = \square$$

T (pointing to the last number story): If you know my rule, what number is

12 * 6?

Invite students to whisper the answer to you or ask them to write it on a slip of paper. Walk around the room and tell the students whether or not they have found the correct answer. Then let someone answer aloud.

S: 12 * 6 = 1.

Record "1" in the box.

In the same manner, ask students to calculate "7 * 7". When many students have found the correct answer (1), let one of them reveal it to the whole class.

T: What is my rule?

S: The result is 1 if the first number is a multiple of the second; the result is 0 if it is not.

T: Since we only consider whether a sentence about numbers is true or false, this rule has to do with a truth table — in particular, a truth table for the relation "is a multiple of".

Write this operation symbol on the board.

$$T_M$$

Erase the number stories and then write this one.

$$9 \sqcup 6 = \square$$

T: What number goes in the box?

S: 18.

L 12

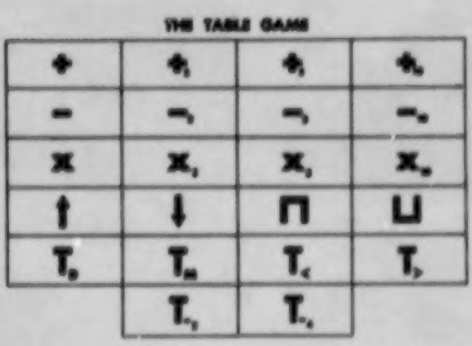
T: How did you get 18?

S: I found the least common multiple of 9 and 6; 18 is a multiple of 9 and a multiple of 6, and 18 is the smallest positive number which is a multiple of both 9 and 6.

Explain the operation yourself if necessary. Write these number stories on the board and complete them as a collective exercise. Answers are in the boxes.

$$5 \sqcup 4 = \boxed{20}$$
$$16 \sqcup 8 = \boxed{16}$$
$$12 \sqcup 15 = \boxed{60}$$

Tape a copy of The Table Game Poster to the board and draw a 9 by 9 table near it.



THE TABLE GAME

$+$	$+_{[illegible]}$	$+_{[illegible]}$	$+_{[illegible]}$
$-$	$-_{[illegible]}$	$-_{[illegible]}$	$-_{[illegible]}$
$\times$	$\times_{[illegible]}$	$\times_{[illegible]}$	$\times_{[illegible]}$
$\uparrow$	$\downarrow$	$\sqcap$	$\sqcup$
$T_{\circ}$	$T_{=}$	$T_{<}$	$T_{>}$
	$T_{[illegible]}$	$T_{[illegible]}$	

*	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

T (pointing to the Table Game Poster): Are there any of these operations you wish to review before we continue?

A brief explanation should suffice if someone wants to review.

Ask these questions to stimulate the students to think about the operations and their tables.*

- Which of these operations has negative numbers in its table? (-)
- Which of these operations has 0 everywhere along this diagonal (trace the diagonal from the upper left corner to the lower right corner) in its table? ($-$, $-_3$, $-_5$, $-_{10}$, $T_<$, $T_>$)
- Which of these operations has 1 everywhere along this diagonal (trace the diagonal from the upper left corner to the lower right corner) in its table? (T_D, $T_=$, $T_{\equiv_3}$, $T_{\equiv_4}$)
- Which of these operations has no 0 entries in its table? ($+$, $\times$, $\uparrow$, $\downarrow$, $\sqcap$, $\sqcup$)
- Which of these operations has numbers larger than 9 in its table? ($+$, $\times$, $\sqcup$)
- Which of these operations has only numbers smaller than 5 in its table? ($+_3$, $+_5$, $-_3$, $-_5$, $\times_3$, $\times_5$, T_D, $T_=$, $T_{\equiv_3}$, $T_{\equiv_4}$, $T_<$, $T_>$)

Exercise 2

As in preceding lessons in this chapter (for example, see Exercise 2 of Lesson L3.2), ask the students to discover an unknown operation $*$ by letting them put numbers in the table. Choose $\times_5$ as the "secret" operation. After each entry is made, tell the class whether or not it is correct. Discuss which of the operations in the list could or could not be $*$ and cross off those that could not. Continue until the table's operation is determined.

Erase the entries in the table and put up a new copy of the Table Game Poster before beginning Exercise 3.

---

* "Their tables" refers to the completed 9 by 9 tables. Completed tables for all twenty-two operations in the Table Game are at the end of this lesson, see pages 127-130.

Exercise 3: The Table Game

Select one student[#] who would be a reliable scorekeeper. Divide the class (except for the scorekeeper) into two teams, Team A and Team B. Distribute individual sheets from the Table Game Analysis Pad to the students.

Play a game involving a table for an unknown operation *. Players determine which of twenty-two given operations (see poster) is *. Although all the plays in the game are made in the same way, the scoring rules divide the game into two parts with different, but related objectives.

The game begins with one or two entries being made in the table as starting clues. Such an entry can be correct or incorrect (indicated by an "×" through it). The players then take time to use the conveyed information to try to eliminate some of the twenty-two possibilities for *, listed on the analysis sheet. The first player from Team A (B) suggests an entry to the table; for example, the player might suggest that 2 * 3 = 6. You make the entry into the table. If correct, tell the class; if incorrect, say "no" and put an "×" through the entry. Allow a few minutes for the players to use the information gained from the play to help identify the operation *. Then give the players the option to turn in their analysis sheets with one of the operations circled. A team gains ten points for each correct answer given by a member. The scorekeeper tallies the points for each team but does not reveal them at this time.

Proceed in the same manner for the second, the third, and the fourth plays except decrease the number of points which can be gained by one each time (10-9-8-7). The last time students may turn in their analysis sheets is after the fourth play is made.

[#] The scorekeeper could also be any adult who is available to you during the class time, e.g. a teacher's aide or a student teacher.

Now the scoring changes. The remainder of the game allows for each of the players who have not had a turn yet to suggest an entry in the table (in the same manner as before). Only now players earn points for their team based on whether or not their entries are correct. The scoring is as follows.

- A team gains one point if a member suggests a correct entry in a square on the diagonal (see shaded squares below).

*	1	2	3	4	5	6	7	8	9
1	▨								
2		▨							
3			▨						
4				▨					
5					▨				
6						▨			
7							▨		
8								▨	
9									▨

- A team gains two points if a member suggests a correct entry in a square other than on the diagonal.

A team is not penalized for an incorrect suggestion; the entry is simply made and crossed out.

The game ends after each player has had a turn. Call on a player to identify * while the scorekeeper computes the scores. We suggest that you give the scorekeeper a tally sheet similar to the one shown in the next illustration. You may wish to reproduce it so that you will have several copies. And, of course, fill in the correct operation for the game presently being played.

L 12

The secret operation is ______.

Points	TEAM A	TEAM B
First part of game		
For correctly identifying ∗		
• after first play 10 pts.		
• after second play 9 pts.		
• after third play 8 pts.		
• after fourth play 7 pts.		
Second part of game		
For correctly making an entry		
• on the diagonal 1 pt.		
• other than on the diagonal 2 pts.		

Play the game several times. These are some possible choices for the unknown operation ∗ with appropriate starting clues.

First game:	Operation $+_4$	Starting clue $2 * 3 = 1$
Second game:	Operation T_{+_4}	Starting clue $8 * 7 = 0$
Third game:	Operation ♦	Starting clue $5 * 5 \neq 0$

The rest of the lesson description describes a possible game. The operation ∗ is $+_4$ and "$2 * 3 = 1$" has been given as a starting clue.

∗	1	2	3	4	5	6	7	8	9
1									
2			1						
3									
4									
5									
6									
7									
8									
9									

T: Knowing that $2 * 3 = 1$ should help you to eliminate some of the possibilities for the operation. Cross out as many operation symbols as you can on your analysis sheet.

A few minutes are allowed for players to individually analyse the situation.

NOTE: For your information only, there are only three possibilities for $*$ which remain: s_e, $\sqcap$, and T_a.

The first player from Team A suggests an entry.

S: $1 * 1 = 1$.

T: That's correct.

*	1	2	3	4	5	6	7	8	9
1	1								
2			1						
3									
4									
5									
6									
7									
8									
9									

A few minutes are allowed for the players to analyse the situation.

T: Does anyone want to turn in their sheets for a possible ten points?

Nobody turns in an analysis sheet.

The first player from Team B suggests an entry.

S: $5 * 6 = 1$.

T: No; that's not correct.

*	1	2	3	4	5	6	7	8	9
1	1								
2			1						
3									
4									
5						X			
6									
7									
8									
9									

A few minutes are allowed for the players to analyse the situation.

L 12

T: Does anyone want to turn in their sheets for a possible nine points?

Several players turn in their sheets. (The operation is actually determined at this point.)

The next two players make suggestions and some more players turn in their sheets for eight points and for seven points.

Student suggests that
4 * 4 = 4 (incorrect).

*	1	2	3	4	5	6	7	8	9
1	1								
2			1						
3									
4				X					
5						X			
6									
7									
8									
9									

Student suggests that
7 * 5 = 0 (correct).

*	1	2	3	4	5	6	7	8	9
1	1								
2			1						
3									
4				X					
5						X			
6									
7					0				
8									
9									

The scorekeeper tallies the points in the game so far but does not reveal them to the class.

TEAM A	TEAM B
41	35

Now the scoring changes. Students suggest entries and earn points for their team based on whether or not the entries are correct.

S: 3 * 5 = 0.

T: That's correct — two points for Team A.

*	1	2	3	4	5	6	7	8	9
1	1								
2			1						
3					0				
4				~~4~~					
5						~~1~~			
6									
7					0				
8									
9									

The game continues until every player has had a turn. The table at the end of the game is shown below.

*	1	2	3	4	5	6	7	8	9
1	1	~~0~~			0				
2	~~1~~		1	~~1~~				1	
3	3	1	4		0				
4	4		~~0~~	~~4~~	~~1~~		~~0~~		
5		0				~~1~~			
6						~~6~~		3	
7		~~2~~			0				
8							1		
9			~~3~~						1

T: While the scorekeeper is computing the scores, who can tell us which operation * is?

S: $\times_5$.

L 12

The scores are revealed.

TEAM A	TEAM B
41	35
2	2
2	2
2	1
1	2
2	2
2	44
52	

NOTE: A variation of this game involves letting the students turn in their analysis sheets for a possible fifteen points after the starting clues are given. The operation in the game sometimes can be determined by only one or two entries. If you choose to play in this way, adjust the scorekeeper's sheet accordingly.

Operation Tables for the Table Game

Completed tables for the twenty-two operations in the Table Game are provided here for your reference.

THE TABLE GAME

$+$	$+_7$	$+_9$	$+_{10}$
$-$	$-_7$	$-_9$	$-_{10}$
$\times$	$\times_7$	$\times_9$	$\times_{10}$
$\uparrow$	$\downarrow$	$\sqcap$	$\sqcup$
T_D	T_M	$T_<$	$T_>$
	T_{-7}	T_{-4}	

↑	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	3	4	5	6	7	8	9
3	3	3	3	4	6	6	7	8	9
4	4	4	4	4	5	6	7	8	9
5	5	5	5	5	5	6	7	8	9
6	6	6	6	6	6	6	7	8	9
7	7	7	7	7	7	7	7	8	9
8	8	8	8	8	8	8	8	8	9
9	9	9	9	9	9	9	9	9	9

↓	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3
4	1	2	3	4	4	4	4	4	4
5	1	2	3	4	5	5	5	5	5
6	1	2	3	4	5	6	6	6	6
7	1	2	3	4	5	6	7	7	7
8	1	2	3	4	5	6	7	8	8
9	1	2	3	4	5	6	7	8	9

⊓	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1
3	1	1	3	1	1	3	1	1	3
4	1	2	1	4	1	2	1	4	1
5	1	1	1	1	5	1	1	1	1
6	1	2	3	2	1	6	1	2	3
7	1	1	1	1	1	1	7	1	1
8	1	2	1	4	1	2	1	8	1
9	1	1	3	1	1	3	1	1	9

⊔	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	6	4	10	6	14	8	18
3	3	6	3	12	15	6	21	24	9
4	4	4	12	4	20	12	28	8	36
5	5	10	15	20	5	30	35	40	45
6	6	6	6	12	30	6	42	24	18
7	7	14	21	28	35	42	7	56	63
8	8	8	24	8	40	24	56	8	72
9	9	18	9	36	45	18	63	72	9

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+	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

+,	1	2	3	4	5	6	7	8	9
1	2	0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0	1	2
3	1	2	0	1	2	0	1	2	0
4	2	0	1	2	0	1	2	0	1
5	0	1	2	0	1	2	0	1	2
6	1	2	0	1	2	0	1	2	0
7	2	0	1	2	0	1	2	0	1
8	0	1	2	0	1	2	0	1	2
9	1	2	0	1	2	0	1	2	0

−	1	2	3	4	5	6	7	8	9
1	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$	$^{-}7$	$^{-}8$
2	1	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$	$^{-}7$
3	2	1	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$
4	3	2	1	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$
5	4	3	2	1	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$
6	5	4	3	2	1	0	$^{-}1$	$^{-}2$	$^{-}3$
7	6	5	4	3	2	1	0	$^{-}1$	$^{-}2$
8	7	6	5	4	3	2	1	0	$^{-}1$
9	8	7	6	5	4	3	2	1	0

−,	1	2	3	4	5	6	7	8	9
1	0	2	1	0	2	1	0	2	1
2	1	0	2	1	0	2	1	0	2
3	2	1	0	2	1	0	2	1	0
4	0	2	1	0	2	1	0	2	1
5	1	0	2	1	0	2	1	0	2
6	2	1	0	2	1	0	2	1	0
7	0	2	1	0	2	1	0	2	1
8	1	0	2	1	0	2	1	0	2
9	2	1	0	2	1	0	2	1	0

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

×,	1	2	3	4	5	6	7	8	9
1	1	2	0	1	2	0	1	2	0
2	2	1	0	2	1	0	2	1	0
3	0	0	0	0	0	0	0	0	0
4	1	2	0	1	2	0	1	2	0
5	2	1	0	2	1	0	2	1	0
6	0	0	0	0	0	0	0	0	0
7	1	2	0	1	2	0	1	2	0
8	2	1	0	2	1	0	2	1	0
9	0	0	0	0	0	0	0	0	0

$+_5$	1	2	3	4	5	6	7	8	9
1	2	3	4	0	1	2	3	4	0
2	3	4	0	1	2	3	4	0	1
3	4	0	1	2	3	4	0	1	2
4	0	1	2	3	4	0	1	2	3
5	1	2	3	4	0	1	2	3	4
6	2	3	4	0	1	2	3	4	0
7	3	4	0	1	2	3	4	0	1
8	4	0	1	2	3	4	0	1	2
9	0	1	2	3	4	0	1	2	3

$+_{10}$	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	0	1
3	4	5	6	7	8	9	0	1	2
4	5	6	7	8	9	0	1	2	3
5	6	7	8	9	0	1	2	3	4
6	7	8	9	0	1	2	3	4	5
7	8	9	0	1	2	3	4	5	6
8	9	0	1	2	3	4	5	6	7
9	0	1	2	3	4	5	6	7	8

$-_5$	1	2	3	4	5	6	7	8	9
1	0	4	3	2	1	0	4	3	2
2	1	0	4	3	2	1	0	4	3
3	2	1	0	4	3	2	1	0	4
4	3	2	1	0	4	3	2	1	0
5	4	3	2	1	0	4	3	2	1
6	0	4	3	2	1	0	4	3	2
7	1	0	4	3	2	1	0	4	3
8	2	1	0	4	3	2	1	0	4
9	3	2	1	0	4	3	2	1	0

$-_{10}$	1	2	3	4	5	6	7	8	9
1	0	9	8	7	6	5	4	3	2
2	1	0	9	8	7	6	5	4	3
3	2	1	0	9	8	7	6	5	4
4	3	2	1	0	9	8	7	6	5
5	4	3	2	1	0	9	8	7	6
6	5	4	3	2	1	0	9	8	7
7	6	5	4	3	2	1	0	9	8
8	7	6	5	4	3	2	1	0	9
9	8	7	6	5	4	3	2	1	0

$\times_5$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	0	1	2	3	4
2	2	4	1	3	0	2	4	1	3
3	3	1	4	2	0	3	1	4	2
4	4	3	2	1	0	4	3	2	1
5	0	0	0	0	0	0	0	0	0
6	1	2	3	4	0	1	2	3	4
7	2	4	1	3	0	2	4	1	3
8	3	1	4	2	0	3	1	4	2
9	4	3	2	1	0	4	3	2	1

$\times_{10}$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	0	2	4	6	8
3	3	6	9	2	5	8	1	4	7
4	4	8	2	6	0	4	8	2	6
5	5	0	5	0	5	0	5	0	5
6	6	2	8	4	0	6	2	8	4
7	7	4	1	8	5	2	9	6	3
8	8	6	4	2	0	8	6	4	2
9	9	8	7	6	5	4	3	2	1

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$T_{\mid}$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0
3	0	0	1	0	0	1	0	0	1
4	0	0	0	1	0	0	0	1	0
5	0	0	0	0	1	0	0	0	0
6	0	0	0	0	0	1	0	0	0
7	0	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	0	1

T_{H}	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0
4	1	1	0	1	0	0	0	0	0
5	1	0	0	0	1	0	0	0	0
6	1	1	1	0	0	1	0	0	0
7	1	0	0	0	0	0	1	0	0
8	1	1	0	1	0	0	0	1	0
9	1	0	1	0	0	0	0	0	1

$T_{<}$	1	2	3	4	5	6	7	8	9
1	0	1	1	1	1	1	1	1	1
2	0	0	1	1	1	1	1	1	1
3	0	0	0	1	1	1	1	1	1
4	0	0	0	0	1	1	1	1	1
5	0	0	0	0	0	1	1	1	1
6	0	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	0	1
9	0	0	0	0	0	0	0	0	0

$T_{>}$	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0
4	1	1	1	0	0	0	0	0	0
5	1	1	1	1	0	0	0	0	0
6	1	1	1	1	1	0	0	0	0
7	1	1	1	1	1	1	0	0	0
8	1	1	1	1	1	1	1	0	0
9	1	1	1	1	1	1	1	1	0

$T_{=_2}$	1	2	3	4	5	6	7	8	9
1	1	0	1	0	1	0	1	0	1
2	0	1	0	1	0	1	0	1	0
3	1	0	1	0	1	0	1	0	1
4	0	1	0	1	0	1	0	1	0
5	1	0	1	0	1	0	1	0	1
6	0	1	0	1	0	1	0	1	0
7	1	0	1	0	1	0	1	0	1
8	0	1	0	1	0	1	0	1	0
9	1	0	1	0	1	0	1	0	1

$T_{=_4}$	1	2	3	4	5	6	7	8	9
1	1	0	0	0	1	0	0	0	1
2	0	1	0	0	0	1	0	0	0
3	0	0	1	0	0	0	1	0	0
4	0	0	0	1	0	0	0	1	0
5	1	0	0	0	1	0	0	0	1
6	0	1	0	0	0	1	0	0	0
7	0	0	1	0	0	0	1	0	0
8	0	0	0	1	0	0	0	1	0
9	1	0	0	0	1	0	0	0	1

APPENDIX: The String Game

The String Game is introduced in Part I (Grade Two) of the CSMP Upper Primary Grades Program and is used in many versions throughout the CSMP Elementary School Program. It gives students an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time it involves them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played using A-Blocks, but later on variations which make use of numbers make their appearance. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks or using numbers, and we begin the process of detailing the various possibilities open to you when you play the game. The different variations of the game described in this appendix are referred to in several of the lessons in this chapter. However, we suggest that you do not restrict your playing of the String Game just to the few lessons in this semester when it has been scheduled. It has been found that it is most beneficial (and enjoyable) for students if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

Equipment

PLAYING BOARD

The equipment can be most easily managed during the game if you have a magnetic[*] chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards are not magnetic, try any portable chalkboard (grid board, etc.) that the school has available. If you do not have a magnetic chalkboard available, you can use your regular chalkboard.

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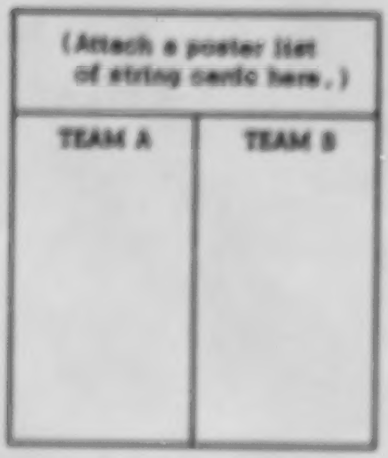
[*] Strictly speaking the chalkboard itself is not magnetic, but rather magnet-sensitive; i.e., magnets adhere to it. However, for simplicity we call such boards "magnetic."

GAME BOARD

The Game Board is divided into three regions all illustrated below.



a) Magnetic : If you do locate a magnetic chalkboard, you will also need a magnetic or metallic Game Board. With a large magnetic chalkboard, say at least three chalkboard squares, the Game Board could be drawn on the board. However, if you have a relatively small (portable) magnetic chalkboard, then you will need to obtain a sheet of metal (minimum size : 60 cm by 60 cm) or to locate a convenient metallic surface in the classroom such as the side of a file cabinet on which to put the Game Board. In this case draw the markings for the three regions of the Game Board on a large sheet of (easel pad) paper and tape this paper to your metallic surface.

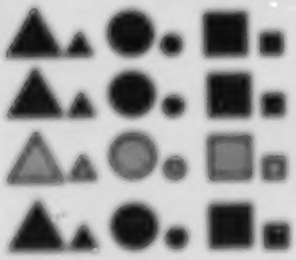
b) Non-magnetic : If you do not have a magnetic chalkboard available for the playing board, then your Game Board could be a large piece of poster board (minimum size : 60 cm by 60 cm). Make the markings for the three regions of the Game Board with a heavy felt-tip pen.

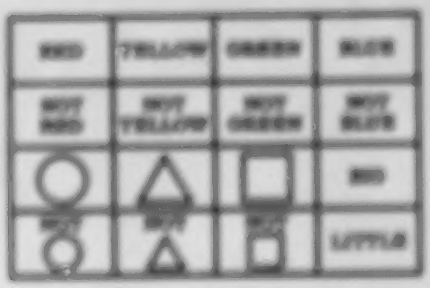
GAME PIECES and STRING CARDS

One set of game pieces and two sets of string cards are needed for each version of the game. Tape a poster list of the string cards to the upper section of the Game Board - it is a constant reminder during the game of the possible labels for the strings.

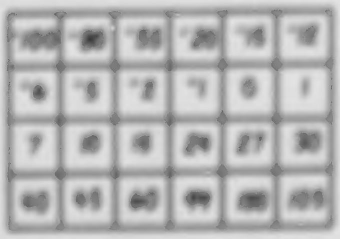
Game Pieces String Cards

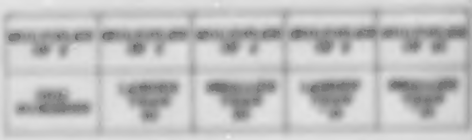
String Game
with
A-Blocks



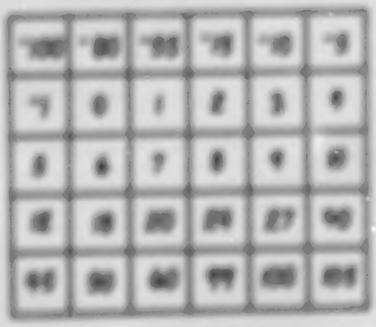

RED
YELLOW
GREEN
BLUE
NOT RED
NOT YELLOW
NOT GREEN
NOT BLUE
BIG
NOT
NOT
NOT
LITTLE

String Game
with
Numbers:
Multiples
and Order


-100
-80
-55
-20
-15
-12
-6
-5
-2
-1
0
1
7
10
18
24
27
30
40
45
60
99
100
105



String Game
with
Numbers:
Multiples,
Divisors,
and Order


-100
-80
-55
-15
-10
-5
-1
0
1
2
3
4
5
6
7
8
9
10
15
18
20
24
27
30
45
50
60
99
100
105

Game pieces and string cards for the String Game with Numbers, all versions, can be found in the Numerical String Game Kit. You will have to note which number cards and string cards to use with the particular version you are playing.

a) Magnetic: The game pieces (A-Blocks or number cards) can be magnetized by sticking to the back of each one a small piece of magnetic material. (Magnetic material is included in the Numerical String Game Kit, and is also available in many hobby and notions departments of stores.) To each string card, stick a small piece of magnetic material to the front of the card, taking care not to obscure what is written thereon.

b) Non-magnetic: Game pieces and string cards can be attached to the Game Board by means of loops of masking tape stuck to the backs. A string card should have a loop of masking tape stuck to the front so that what is written on the card is not obscured. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if a loop of tape loses its stickiness, it can be replaced on the spot. An alternative is to use a small wad of a plastic caulking compound (Rope Caulk or Mortite, for example) in place of the loop of masking tape. This material will maintain its stickiness over a much longer period of time.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the playing board using two (or three) different colors. Next to each of these strings attach one string card face-down. Stand the Game Board on the chalk tray, or have it near and convenient to the playing board. Share the game pieces between the two sections of the Game Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; rows 1, 2, 3 vs. rows 4, 5, 6; etc. Call one team "Team A" and the other "Team B". Before any student takes a turn, correctly place an equal number (at least one) of each team's game pieces in the string picture. This reserves from the first players the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by the number of pieces you place on the game board before the game begins.

Object of the Game.

Each team tries to place all of its game pieces correctly (according to the face-down string cards) in the string picture. The winning team is the one which is first to place all of its game pieces correctly and which identifies the face-down cards correctly, after playing according to the rules.

Rules of the Game.

1) The game is played in silence. This is to avoid one student telling another where to play a certain game piece, or what the strings are for. Each student should have the opportunity to analyze the game alone. Infringement of this rule by anyone is penalized by the talker's team losing its next turn.

2) The teams alternate, and the members take turns within each team. A player comes to the board and selects a piece from his or her team's collection to place in one of the regions of the string picture.

3) You are the judge. If the piece is correctly placed, say "yes." The piece remains in the string picture and the player immediately has a second turn (no player may have more than two consecutive turns). If the piece is incorrectly placed, say "no." The player returns the piece to the team's unplayed collection and play passes to the other team.

As an aid to judging, prepare a "crib-sheet" showing the correct position of each game piece or at least reminding you of what is on the face-down cards. If at any time you discover that you have made an error, say so immediately and rectify the mistake. Either move an incorrectly placed piece whose position you had approved to its correct region or replace in the string picture a correctly placed piece that had been rejected because you had disapproved its position.

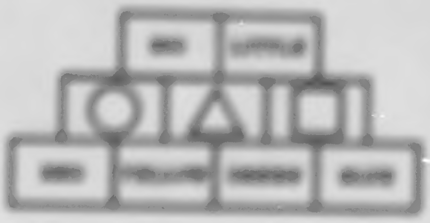
4) When a team has correctly placed all of its pieces, the player who placed the last piece may thereupon attempt to identify each of the string cards.

If these are all correct, the team has won. If a mistake is made (even if it is only in the name of one of the string cards), simply indicate that the identification is incorrect and let the game continue.

5) If a team has exhausted its stock of game pieces, and the strings have not been identified, that team continues on its turn to attempt to identify the strings, while the other team works to place its game pieces.

Version A: The String Game with A-Blocks

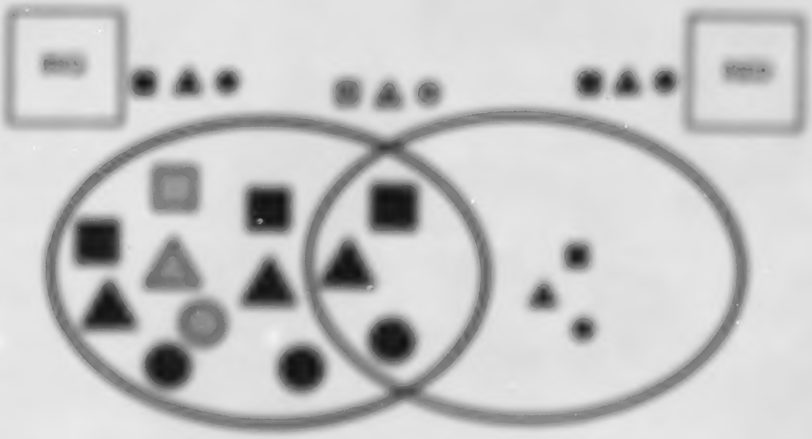
This simplest version of the game uses twenty-four A-Blocks as game pieces and only nine string cards.



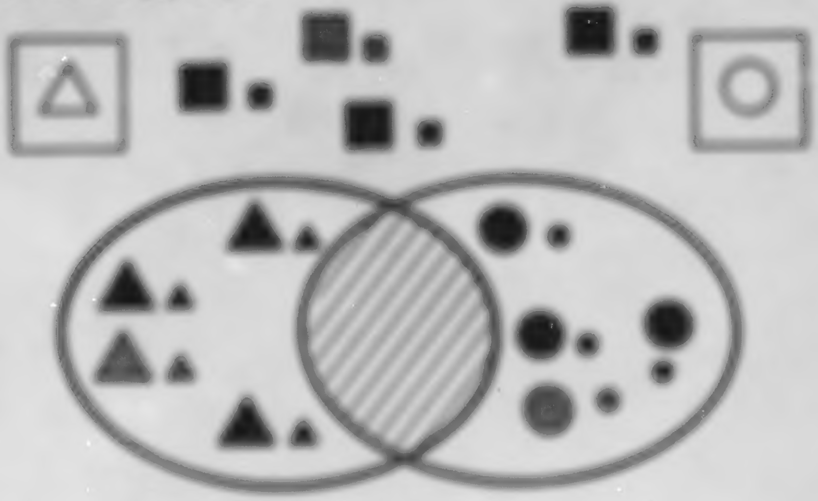
We provide here several example "crib-sheets" for variations of the game with two and with three strings.

Variation B: Two Strings

Example 1: No Empty Regions

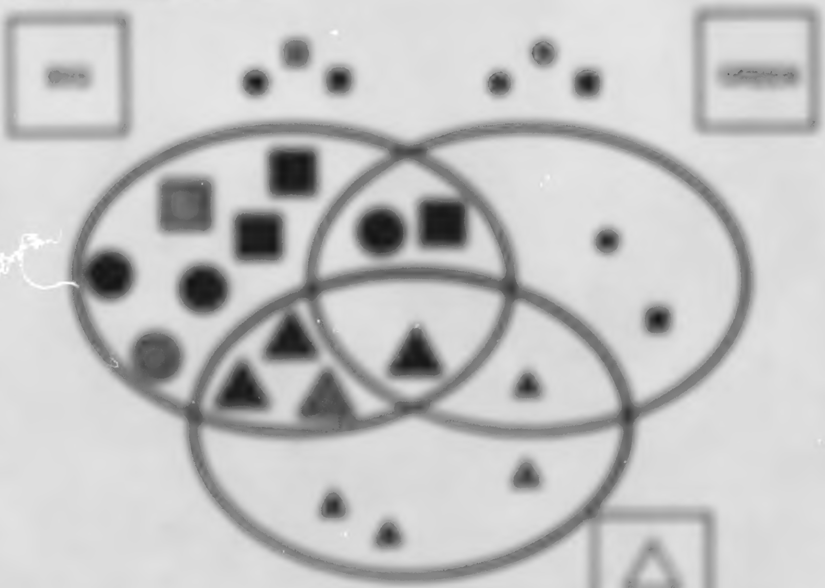


Example [illegible]: One Empty Region

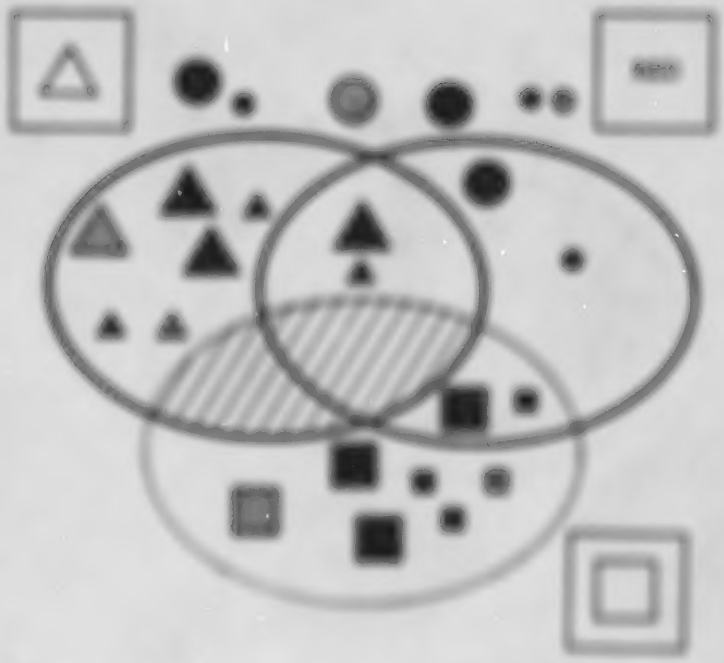


Variation [illegible]: Three Circles

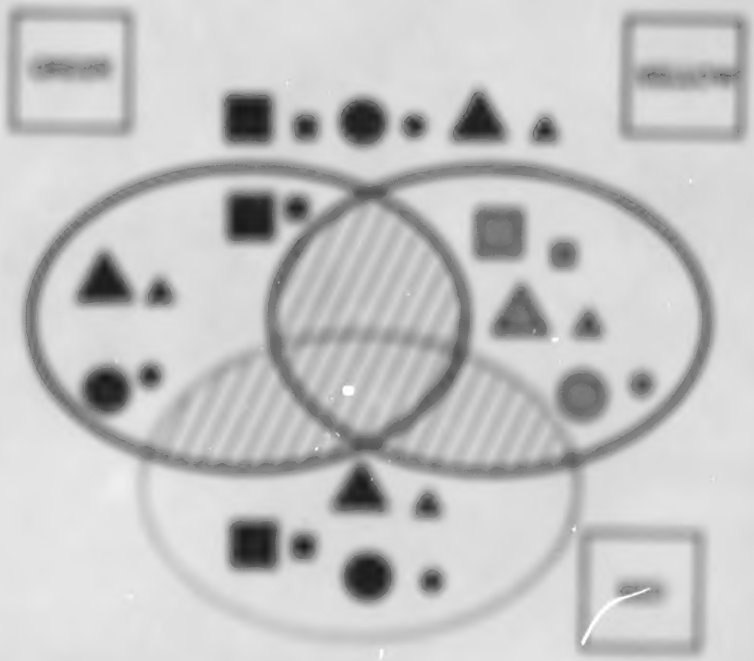
Example 1: No Empty Regions



Example 2: Two Empty Regions



Example 3: Four Empty Regions



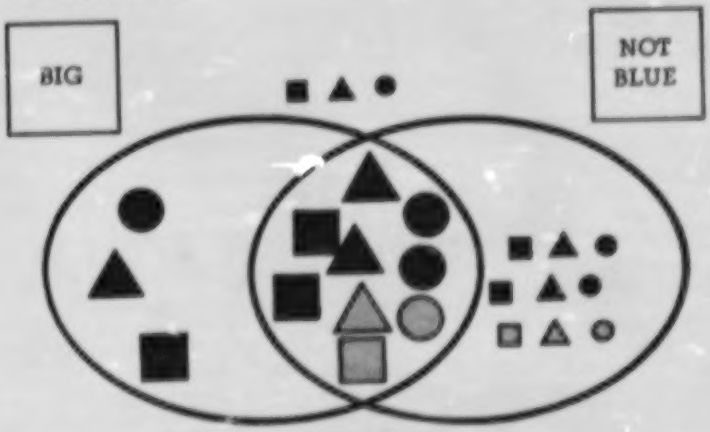
Version B: The String Game with A-Blocks, using "Not" Cards

A more complicated version of the game with A-Blocks uses all sixteen of the string cards. (See the description under "Equipment" of game pieces and the string cards needed for the game with A-Blocks.)

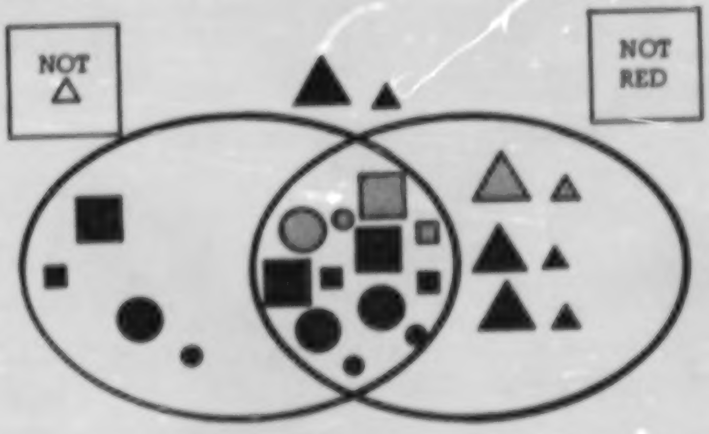
NOTE: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly for "NOT LITTLE" and "BIG"), such a "not" card is redundant. If a student questions why we do not include these possibilities, let the class discuss that it is unnecessary.

We provide here several example "crib-sheets" for this version of the String Game played with two strings.

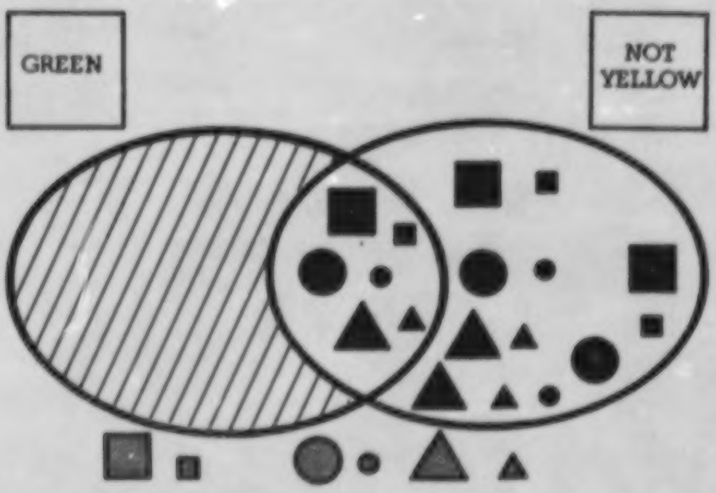
Example 1: No Empty Regions



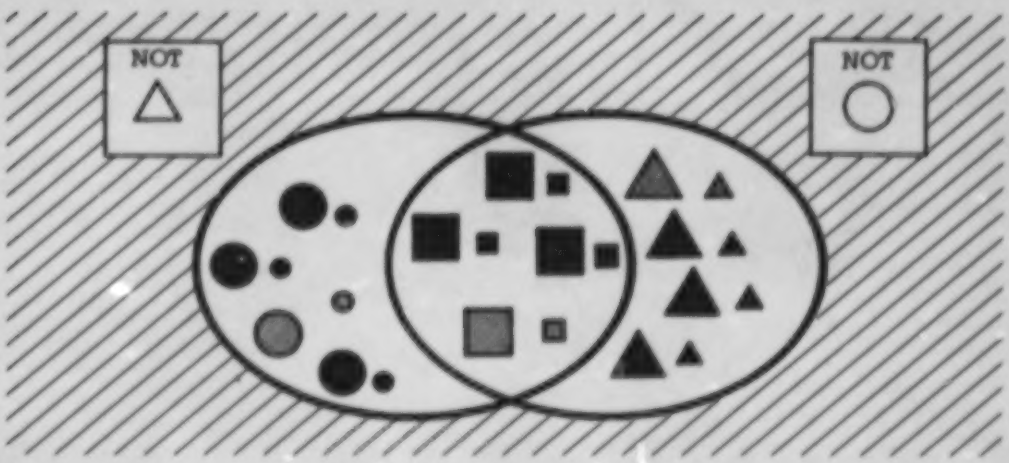
Example 2: No Empty Regions



Example 3 : One Empty Region



Example 4 : One Empty Region



Of course, by hatching the "outside" region of this diagram we mean to indicate that no game pieces can be placed there correctly. Strictly speaking, that region is not empty because, for example, the number 50 is in the outside region.

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your students.

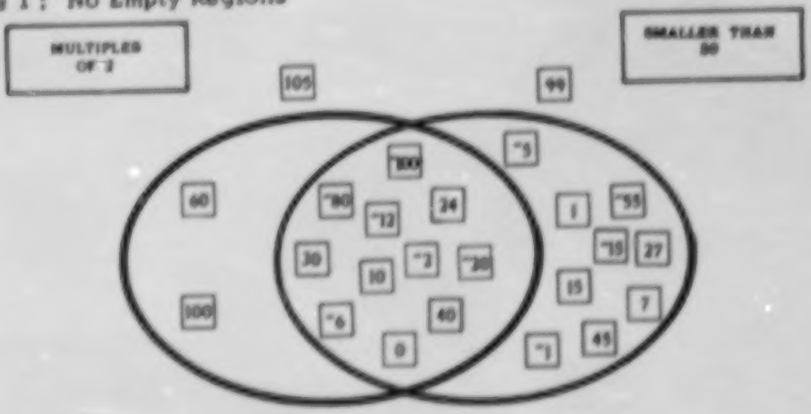
Version C : The String Game with Numbers : Multiples and Order

This version of the game uses twenty-four number cards as game pieces and ten string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string.

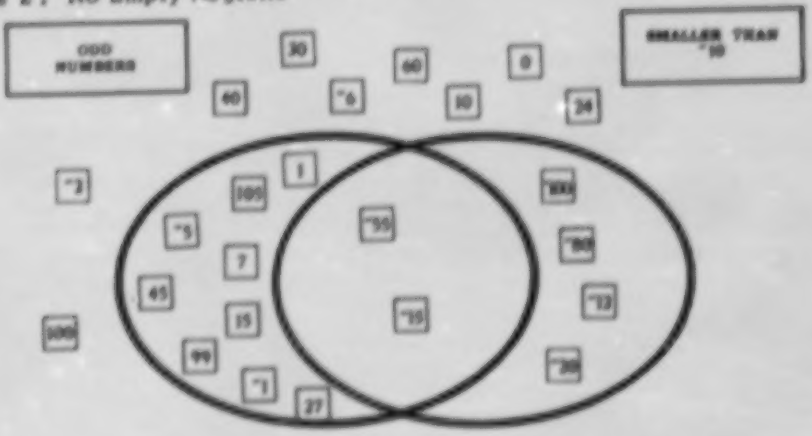
We provide here several example "crib-sheets" for variations of this game using two and three strings.

Variation #1 : Two Strings

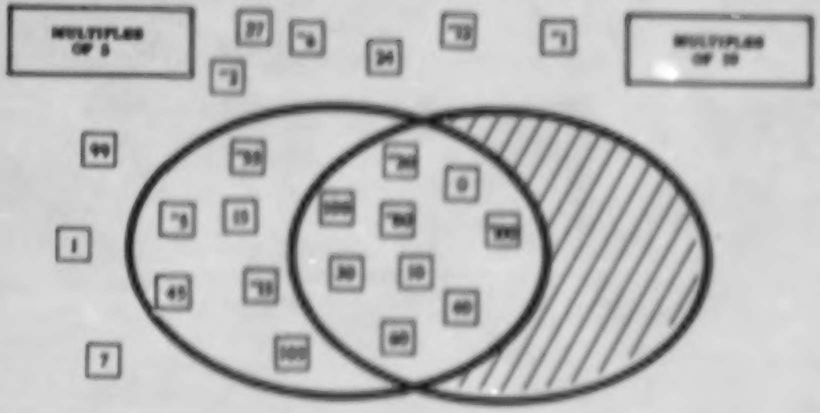
Example 1 : No Empty Regions


MULTIPLES OF 2
SMALLER THAN 50
105
99
$^{-}5$
$^{-}100$
60
$^{-}80$
$^{-}12$
24
1
$^{-}55$
30
10
$^{-}2$
$^{-}20$
$^{-}15$
27
15
100
$^{-}6$
40
0
$^{-}1$
45
7

Example 2 : No Empty Regions


ODD NUMBERS
SMALLER THAN $^{-}10$
30
60
0
40
$^{-}6$
10
24
$^{-}2$
1
105
$^{-}100$
$^{-}5$
$^{-}55$
7
$^{-}80$
45
15
$^{-}15$
$^{-}12$
100
99
$^{-}1$
27
$^{-}20$

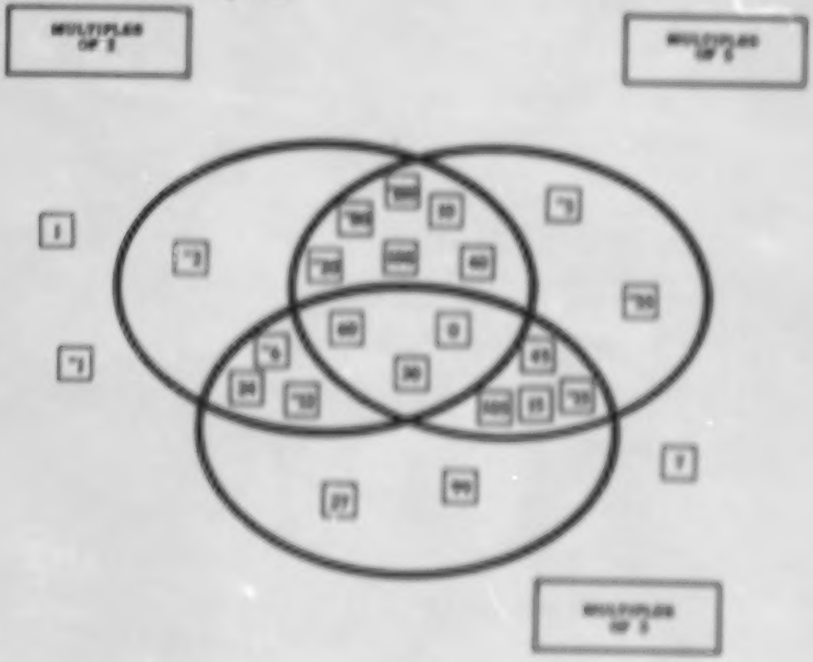
Example 3 : One Empty Region


MULTIPLES OF 5
MULTIPLES OF 10

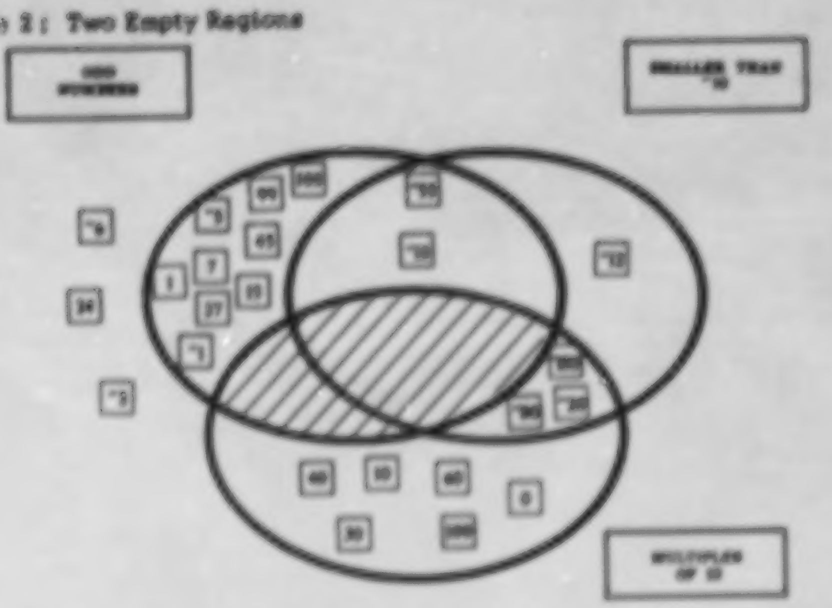
Variation #2 : Three Strings

CAUTION : The jump in difficulty from the two-string variation to the three-string variation is much greater in the case of Version C of the String Game than it was in the case of Version A. Hence you should not expect to proceed to Variation #2 in this case as rapidly as you might have done with Version A.

Example 1 : No Empty Regions


MULTIPLES OF 2
MULTIPLES OF 5
MULTIPLES OF 3

Example 2: Two Empty Regions

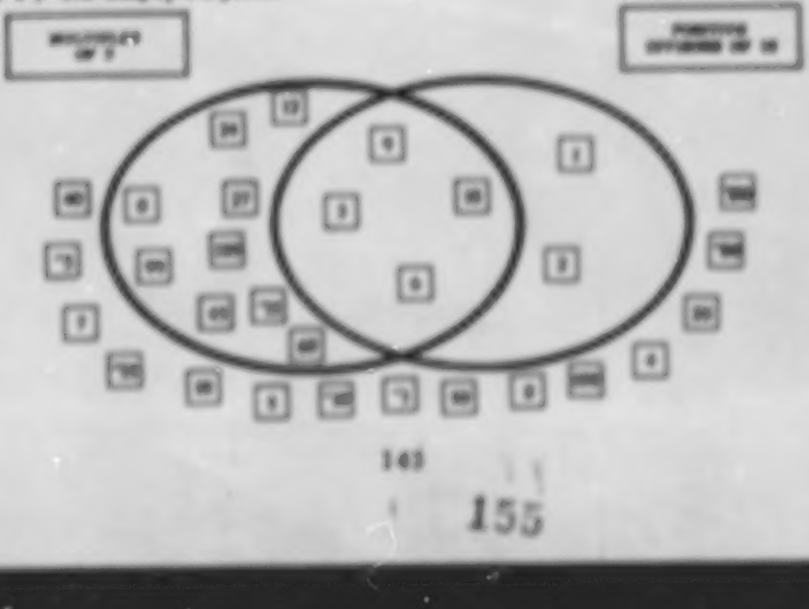

ODD NUMBERS
SMALLER THAN 10
MULTIPLES OF 10

Version D: The String Game with Numbers: Multiples, Divisors, and Order

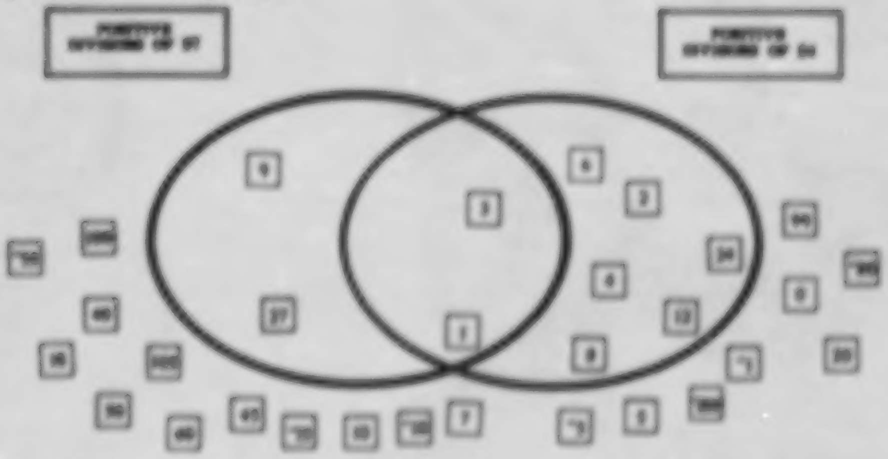
This version of the game uses thirty number cards as game pieces and sixteen string cards as described under "Equipment". The equipment for this game is included in the Numerical String Game Kit. Note that in the Kit three sets of string cards are included. This allows you to give the same label to more than one string. Two examples of this are shown in "crib-sheet" examples provided here; however, before using such a game the players should have previously understood that this was a possibility.

Variation #1: Two Strings

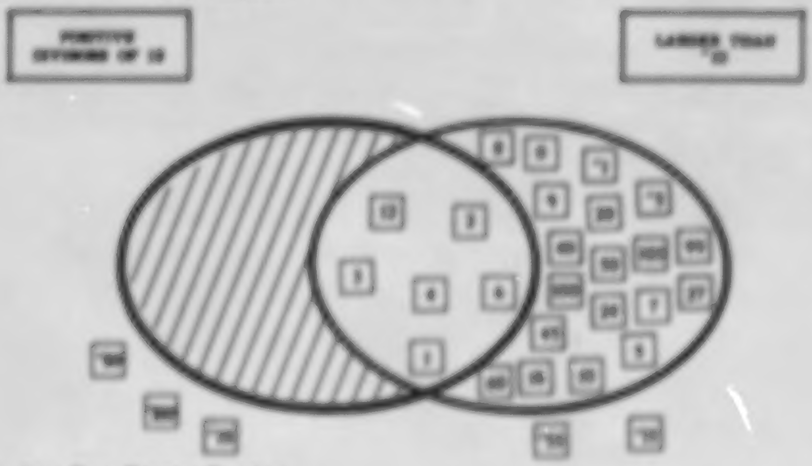
Example 1: No Empty Regions


MULTIPLES OF 3
POSITIVE DIVISORS OF 18

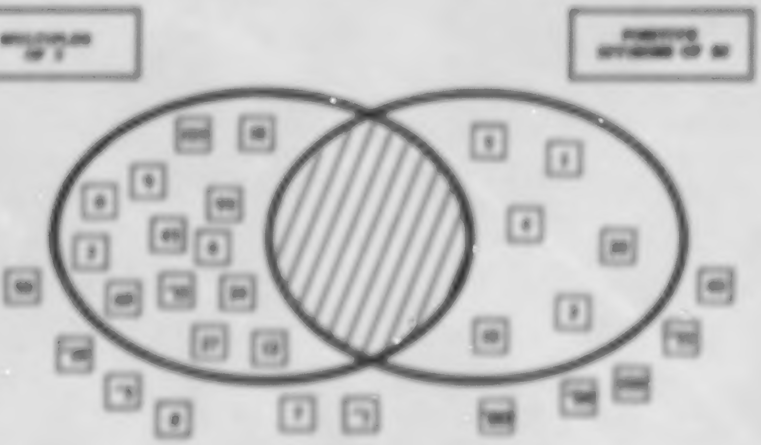
Example 2: No Empty Regions



Example 3: One Empty Region



Example 4: One Empty Region



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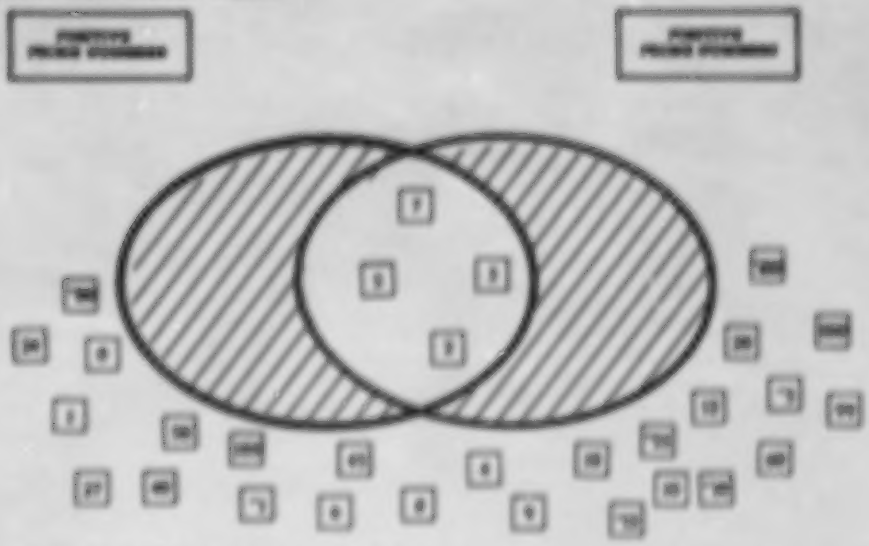
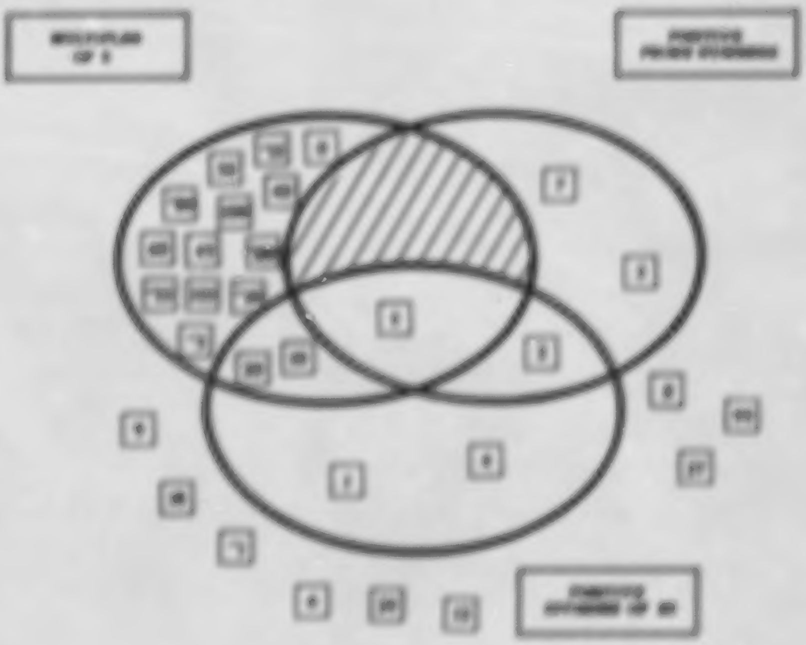
Example 5: Two Empty Regions



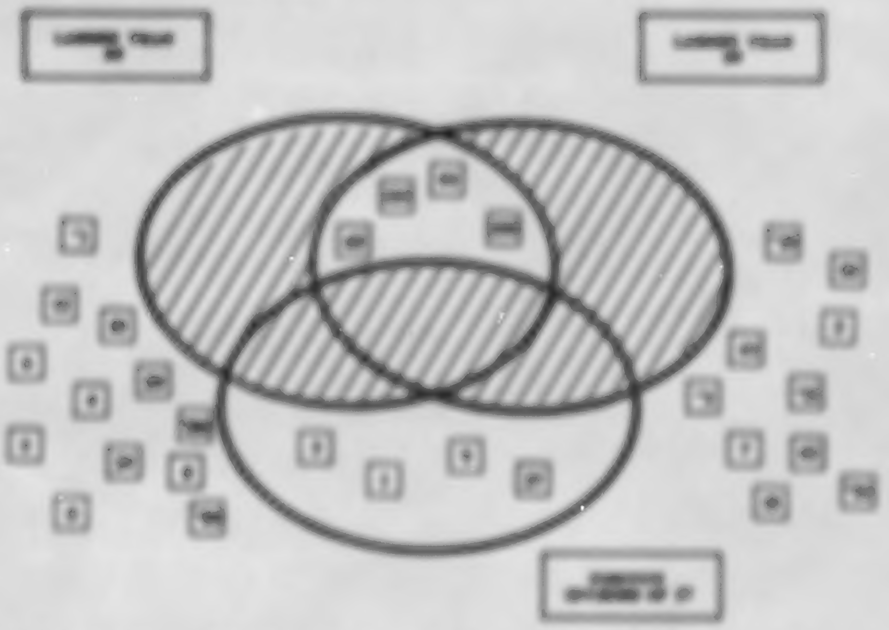
__Variation #2: Three Strings__

The warning concerning the jump from the two-string variation to the three-string variation on page 142 applies even more so to Version D.

Example 1: One Empty Region



Example 3: Five Empty Regions



Version 3: The String Game with Special Scoring Rules

In this version of the game special scoring rules to determine the winning team are introduced. Both the String Game with numbers and with shapes can be played with these scoring rules; however, in the lessons of this book only the Numerical String Game is played this way.

Scoring

Each student is given a list of the possible string labels on a sheet from the Individual String Game Analysis Pad. One round of the game is completed by having one player from each team take a turn. After each round the students have an opportunity to privately identify the face-down string cards. They do this by circling on their RED and BLUE lists what they believe is on the hidden cards and submitting it to the judge (teacher). However, each student has just one chance to make a private identification. If a player correctly identifies the string cards after a particular round, then points (according to the following schedule) are added to the player's team score.

Round 1 = 200 points
Round 2 = 150 points
Round 3 = 100 points
Round 4 = 75 points
Round 5 = 50 points
Round 6 = 25 points

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After Round 6, play out the game without further private identifications. No more additional points can be scored by a team until the end of the game. The team that correctly identifies the string cards at the end of the game scores an additional 100 points plus 5 points for each unplayed game piece the opposing team has left on the Game Board. The winning team is the one with the greater total score.

Comments

1) Allow about two to three minutes before the start of Round 1 for students to analyze the starting clues (correctly placed game pieces).

2) Keep count of the rounds played on the board so that both you and the students know which round has just been completed.

3) After each round give the students time (perhaps 15-30 seconds) to study the picture, to mark their lists, and if desired to submit them.

4) When the students submit their lists, write on each one the round after which it was turned in and sort them by teams. Do not return them to the students. You can total the team scores at the end of the game. Do not tell the students that their identifications are correct or incorrect, but encourage them to write their findings on a separate sheet of paper so as not to make mistakes when they play. They should continue to check their identifications during the remainder of the game.

5) It may be even more important to insist on silence during this kind of game because an individual can have more influence on a team's winning or losing. We suggest you be strict about penalties for talking during the game.

Footnote

Scoring rules should not be imposed on the String Game until students have become very familiar and experienced in the play of the game. Many students already will be playing strategically, analyzing the clues given by correctly played game pieces. Others may still be playing in a random way. The introduction of scoring rules obviously will favor the better students and perhaps challenge them in a way the usual game is no longer doing. It provides them the opportunity to more dramatically affect their team's chances of winning. For many students the scoring rules may provide incentive to begin analyzing the game and thinking about what information is given by a correctly played game piece.

We do not suggest that you play the String Game very often with the scoring rules. They do impose complication and give perhaps too much advantage to the better players. After playing the game with scoring rules a couple times, you should find that many students have become significantly better players. Returning to the usual way of playing the game, most students will continue to be better players.

One advantage of using scoring rules when you play the String Game and have only a small amount of time is that you can stop the game before the end and have a reasonable way to determine the winning team. The winning team should be the one with the larger total number of points from identification of string cards.

GEOMETRY AND MEASUREMENT

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades
Part V

Geometry and Measurement

CEMREL, Inc.

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Worksheets [illegible] . . . . . [illegible]
1.9 [illegible] a Block (II) . . . . . [illegible]
Worksheets [illegible] . . . . . [illegible]
1.10 [illegible] a Block (III) . . . . . [illegible]
Worksheet [illegible] . . . . . . [illegible]
1.11 [illegible] a Block (IV) . . . . . [illegible]

CHAPTER 2. [illegible] . . . . . . [illegible]
2.1 [illegible] . . . . . . [illegible]

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[illegible]

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Lesson Seven [illegible] — A [illegible] Game [illegible]
Lesson Eight [illegible] — Triangle Constructions [illegible]
Lesson Nine [illegible] — Breaking a Stick (1) [illegible]
Lesson Ten [illegible] — Breaking a Stick (2) [illegible]
Lesson Eleven [illegible] — Breaking a Stick (3) [illegible]
Lesson Twelve [illegible] — Breaking a Stick (4) [illegible]

GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense.

A benefit of early development of geometric thinking is that it provides alternative opportunities for "slow learners." Children develop number concepts at individual rates. Relative lateness of a child's numerical development often signifies a difficulty in remembering number facts. A common response to this problem is assignment of arithmetic drill, which often serves to confirm the diagnosis of slowness. However, numerical development may not be an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

The lessons of this strand provide an opportunity to explore geometric notions informally. The activities are varied to meet other areas of the program and are diffused over a long period of time. As a natural consequence of their involvement in the activities, the students will develop knowledge and skills.

But the effects of such an informal program must be measured in their appearance in subsequent contexts. No goals for mastery can be posted here; none could be certified. The focus of this strand is experience.

1

GEOMETRIC CONSTRUCTIONS

Most of our geometry so far has been concerned with points and lines. Parallelism was the central concept of our study of points and lines. Through a variety of informal experiences, students developed their geometric intuition of points and lines and their relations in constructions based on parallelograms. The circle is another fundamental geometric shape. The lessons of this chapter provide experience with circles and their relations with points and lines. Through use of the translator, students were able to perform accurate constructions. Care and precision in drawing circles through use of a compass must equal that invested in lines. The compass is an ancient geometric tool with a tradition antedating the earliest works on geometry. Its power in geometric study increases with the sophistication of the student.

The first activities reinforce the concepts of center and radius of a circle. A series of constructions present an opportunity to practice using a compass and develop a sense of the dependence among the radii of various circles. While the lessons on parallelism provided several applications related to other strands of the curriculum, especially in uses of the number line construction, no direct comparisons of lengths of line segments were possible, unless they were parallel. The compass provides a tool for such comparisons. The activities proceed with the classical construction of the midpoint of a line segment and of perpendicular line segments, first introduced for constructions by use of a folded paper square corner. As in their work with the translator, the students' dynamic interaction with the compass provides a basis for the growth of spatial intuition and an understanding of informal geometric relationships.

Next follow a series of activities based on constructions of polygons under restrictive conditions. The ultimate objective is to bring the student to some realization of a relationship between the lengths of given line segments and the number and type of shapes that can be constructed from them. The simple construction of duplicating a given line segment leads to the problem of constructing polygons with a specified number of sides, all of the same length.

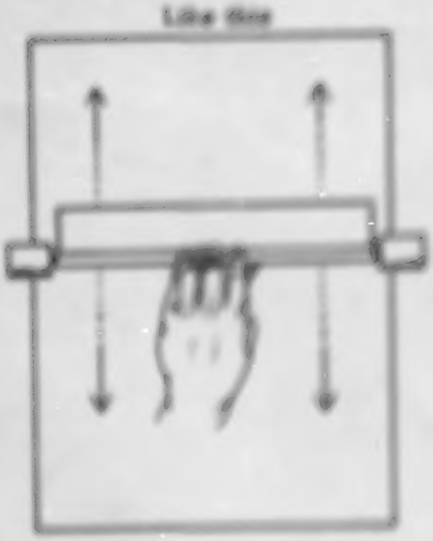
Then two line segments are given with which to construct quadrilaterals. Discussion of the contrast between the small number of essentially different solution shapes and the infinite variety of orientations and areas that can occur supports the students' developing sense of space. Constructing triangles with two given line segments leads to a recognition that the relative lengths of the line segments control the result of one or two essentially different solutions. In this way the students discover a special case of the triangle inequality in a context in which the alternatives are both productive, rather than in the general case in which the alternative solutions are one or none. Of course, this realization is followed by the problem of constructing a triangle using three given segments, which leads to a formulation of the triangle inequality itself.

In the final lessons, a problem is posed: If a stick is randomly broken at two points, what is the probability that the three pieces will form a triangle. In studying this problem, students will have the opportunity to relate their work in several strands of the curriculum, numerical and probabilistic as well as geometrical.

NOTE ON TRANSLATORS

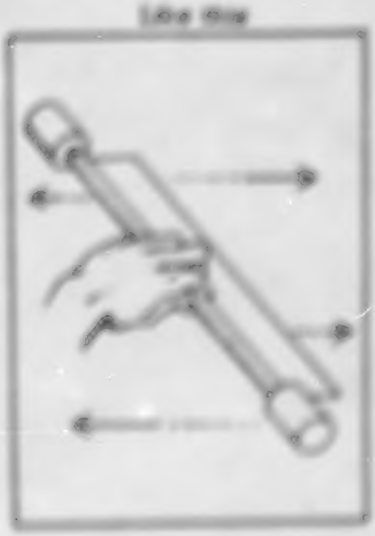
Before using the translator in the classroom, practice using it yourself. Take some time to practice using a student translator as well as the demonstration translator. Occasionally, a construction at the board requires an odd posture. In introducing the class to the translator, mention the sharp corners of the metal blade as a warning against misuse. Provide a few small pieces of masking tape to each student so that paper can be anchored to the desk. This precaution will eliminate frustrating slippage of the paper while a student is drawing. Instruct the class in the proper manipulation of the translator. One should hold the plastic tube at the center and gently push or pull to move it across the paper or desk. The pictures on the next page illustrate proper and improper handling.

Like this



or

Like this



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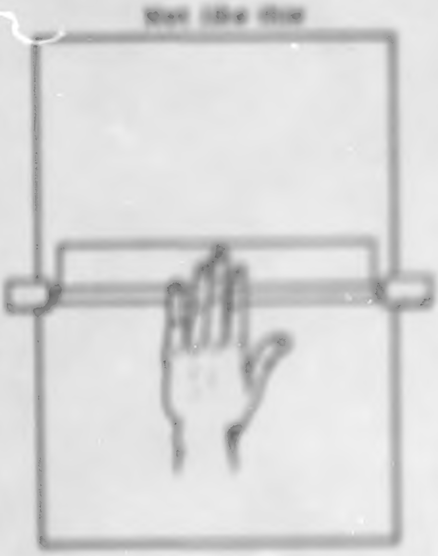
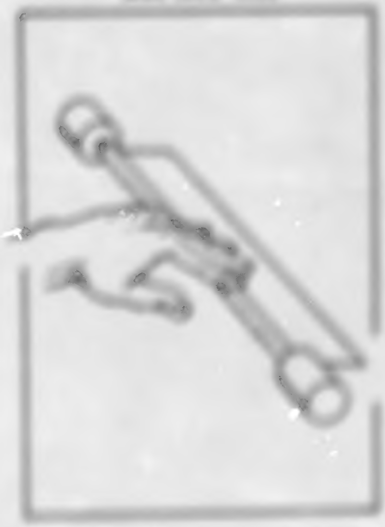


and

Not like this



That the blade is not rigidly attached to the plastic tube should cause no problem when a slow steady pressure is used in rolling the translator. The advantage of this means of attachment is that blades may be easily repaired in the event of breakage. For repairs, simply provide yourself with a roll of strapping tape as used in the construction of the translator. However, we anticipate that repairs will only rarely be necessary.

CAPSULE LESSON SYNOPSES

Section 1.1: Review some of the language associated with circles and the use of a compass for constructing circles. Practice using the compass by drawing many circles to some specification. Make some observations about the resulting drawings and find relations among the resulting radii.

Section 1.2: Review the circle constructions of Section 1.1 and present two more similar constructions. Again make observations about the resulting drawings and find relations among the resulting radii. Compare the four drawings. Construct a lattice by connecting the centers of a family of circles drawn to a certain specification.

Section 1.3: Review the notion of a square corner and produce one by folding paper. Use a square corner to construct line segments perpendicular to given line segments and through given points. Construct shapes with all square corners.
(Worksheets G01, G02)

Section 1.4: Use a compass to find the midpoint of a line segment and observe that the construction yields also a line segment perpendicular to the given line segment. Develop methods of using the compass to construct perpendicular lines to given lines and through given points.
(Worksheets G03, G04, G05)

Section 1.5: Use a compass and a straightedge to duplicate line segments. Discuss a variety of polygons and then use a compass and a straightedge to construct equilateral, eight-sided polygons.

Section 1.6: Use a compass and a straightedge to construct an equilateral, six-sided polygon; an equilateral, four-sided polygon; and a regular (equilateral) triangle. Given two line segments, use a compass and a straightedge to construct as many essentially different four-sided shapes as possible with each side the same length as one of the given line segments. Make some observations concerning the variety of shapes that can be constructed and the possible diagonal length of various of the shapes.
(Worksheet 206)

Section 1.7: Given two line segments, construct as many essentially different triangles as possible with each side the same length as one of the line segments. Discover a rule relating the length of the segments to the number of different triangles. Repeat this activity with three given line segments and discover the triangle inequality. Pose the problem of breaking a stick into three pieces and deciding whether a triangle can be formed with the three pieces.
(Worksheets 207, 208)

Section 1.8: Introduce the problem of breaking a stick into three pieces and trying to form a triangle. Examine many cases to decide the conditions for success and failure. Rephrase the triangle inequality.
(Worksheets 209, 210)

Section 1.9: Review the problem of breaking a stick introduced in Section 1.8 and decide that a method of choosing two breaking points at random must be found. Using parallel projections (with a transformer), establish a means for making this random choice.
(Worksheets 211, 212)

Section 1.10: Review the double projection method (from Section 1.6) for associating two points on a line with one point above the line. In the context of the problem of breaking a stick, construct a "honey" triangle with one side on the line and the third vertex found by associating the end points of the side with a point above the line. Find that only points inside the triangle project onto points of the side of the triangle on the original line. Practice the double projection with points inside the "honey" triangle and try to construct triangles with the three resulting pieces of the stick.
(Worksheet II-3)

Section 1.11: Recall the problem of breaking a stick as developed in Sections 1.4, 1.8, and 1.10. Apply the double projection technique and the triangle inequality to solve the problem.
(Worksheet II-4)

[illegible] Compass Constructions A

CAPSULE LESSON SUMMARY

Review some of the language associated with circles and the use of a compass for constructing circles. Practice using the compass in drawing many circles to given specifications. Make some observations about the resulting drawings and [illegible] the resulting [illegible].

MATERIALS

Teacher: Chalkboard compass; colored chalk; straightedge
Student: Paper; colored pencils; individual compass; straightedge

DESCRIPTION OF LESSON

In this lesson students will again meet concepts related to circles: center and radius. The focus of the lesson is on practice in using a compass to draw circles carefully and accurately, a skill which will be important in the later lessons of this series. Students' compasses should be of good quality. The axis of the legs should be tight enough to prevent slippage as a circle is being drawn, but not so tight as to make setting radii difficult. A sharp pencil in the compass will aid accuracy. Placing a piece of cardboard or a section of a newspaper beneath the paper on which a drawing is being made will help to prevent the compass point from slipping on the desk.

Exercise 1

Make a red dot in the center of the board. Using this red dot as center, draw a circle on the board with radius about 30 cm.

G 1

T: Here is a circle. The red dot is special with respect to the circle and has a name. What is it called?

S: The center of the circle.

T: Yes, and we also have a name for the distance from the center to any point on the circle. What is that distance called?

S: The radius of the circle.

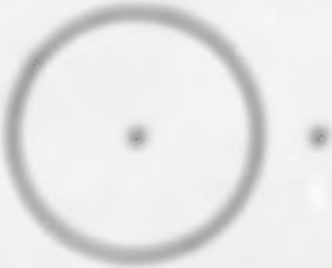
Display your compass with the radius unchanged after having drawn the circle.

T: Look at the compass with which I drew this circle. What is the distance between the point of the compass and the chalk?

S: It is the same as the radius of the circle.

T: Yes. We call that distance the radius of the compass. Of course, we can easily change the radius of the compass by moving the legs closer together or further apart.

Add a blue dot to your picture about 15 cm outside the circle and to one side.



T: Let's draw some more circles on the board. The rule is that the center of each new circle must be on the original circle and the blue dot must be part of each new circle.

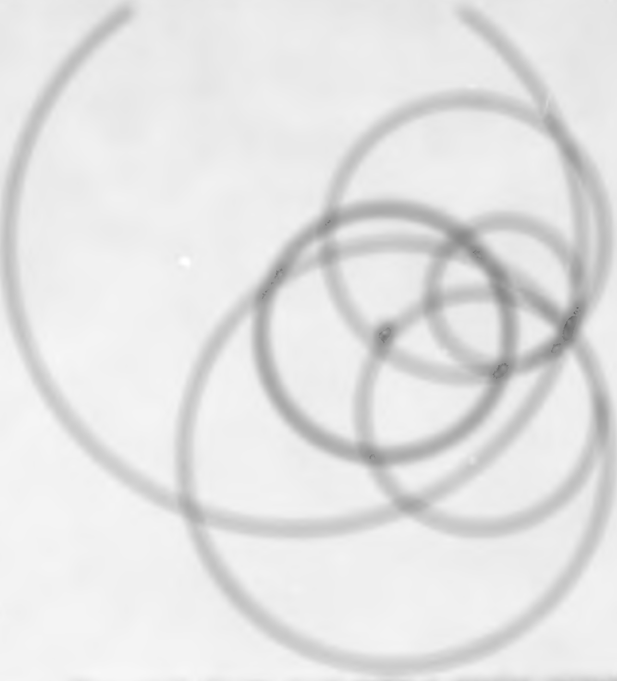
Invite several students to add circles to the drawing and encourage discussion of the emerging picture.

T: Can we use any point as the center of a new circle?

S: The centers of new circles must be on the original circle.

T: Can any point of the original circle be the center of a new circle?

Let any skeptical student choose a point on the original circle. Then invite another student to construct a new circle according to the rule. When four or five circles have been drawn, the class will be aware of the technique of locating a center and then setting the radius of the compass by placing its point on the prospective center and its lead on the blue dot.



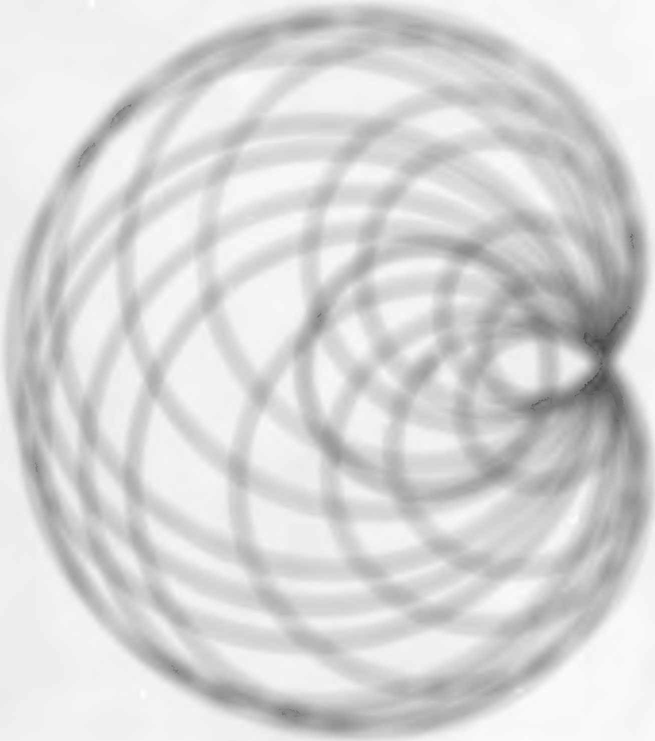
T: Now each of you must make a similar picture. Mark a red dot in the center of your paper. Draw a medium sized circle and mark a blue dot outside the circle. Then draw more circles according to the rule: the center of each new circle is on the original circle and the blue dot is part of each new circle. Make the red and blue dots small and draw your circles carefully and accurately.

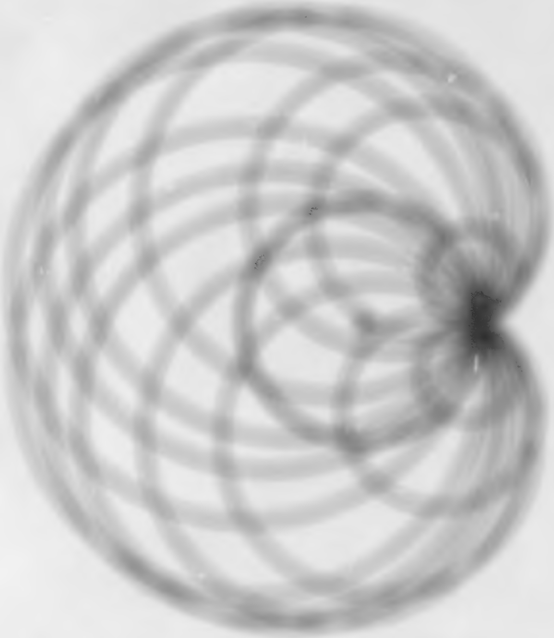
G 1

[illegible]

[illegible]

[illegible]



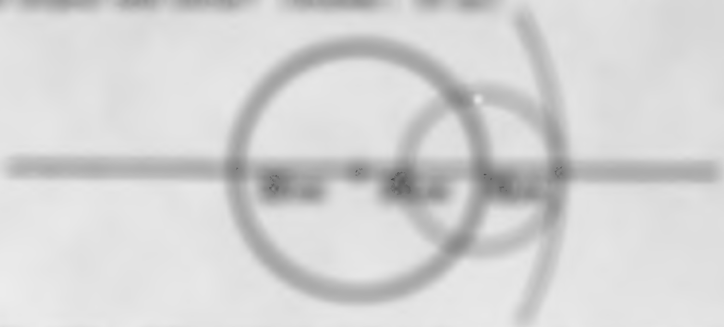


[illegible]

- [illegible]
- [illegible]
- [illegible]

G1

- [illegible]



- [illegible]

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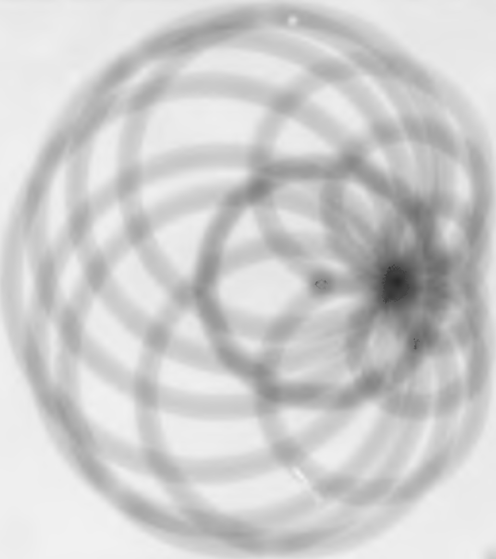
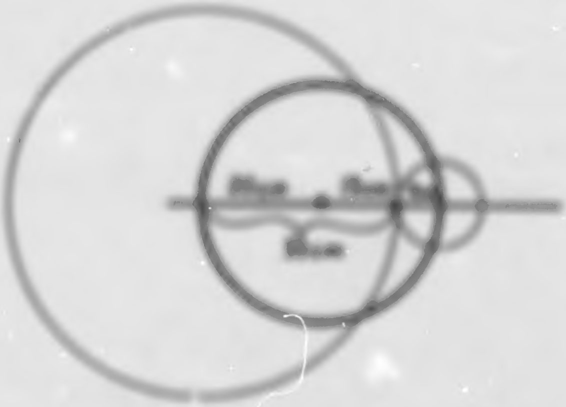


G2

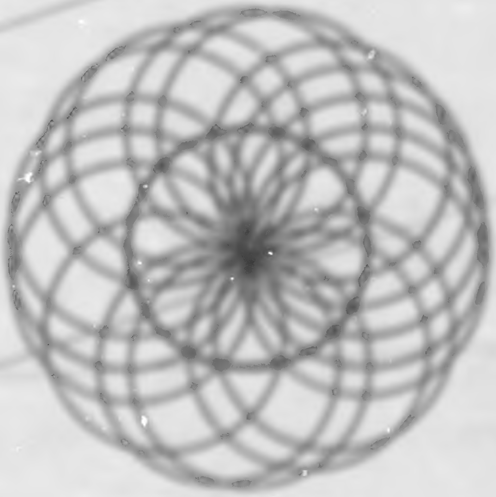
As at the end of Lesson G1.1, lead a discussion of the collection of circles in this picture. Raise similar questions relating the radii of the new circles.

- Where is the center of the circle with the shortest radius?
  (Answer: on the line through the red dot and the blue dot and closest to the blue dot)
- Where is the center of the circle with the longest radius?
  (Answer: on the line through the red dot and the blue dot and closest to the red dot)
- How many circles can be drawn with each possible radius?
  (Answer: one each for the shortest and the longest radii and two each for all other possible radii)
- If the radius of the original circle is 20 cm and the radius of the smallest new circle is 7 cm, what is the radius of the longest new circle?
  (Answer: 13 cm)



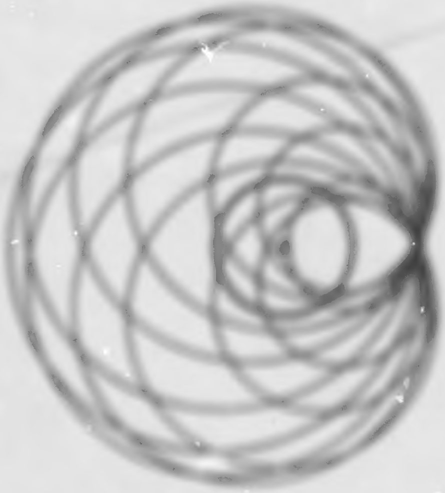
T: Suppose we start again, but this time put the blue dot at the center of the original circle; that is, the red and blue dots are at the same point.

Begin the drawing at the board and invite students to add circles. The class should quickly recognize that all of the new circles have the same radius and in fact have the same radius as the original circle.

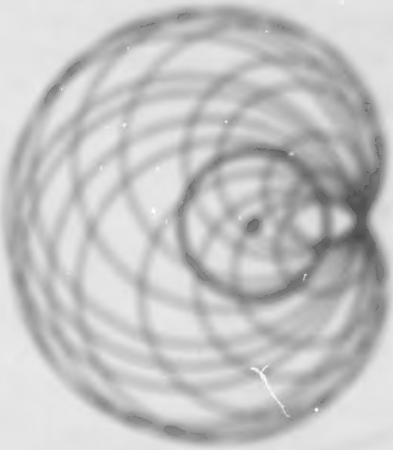


A group project can be made of producing a series of similar pictures. All original circles should be the same size and located in the same position on the paper. The blue dot should be shifted slightly from picture to picture. The resulting sequence of pictures will emphasize the effect of the distance between the red and blue dots on the configuration and sizes of the added circles.

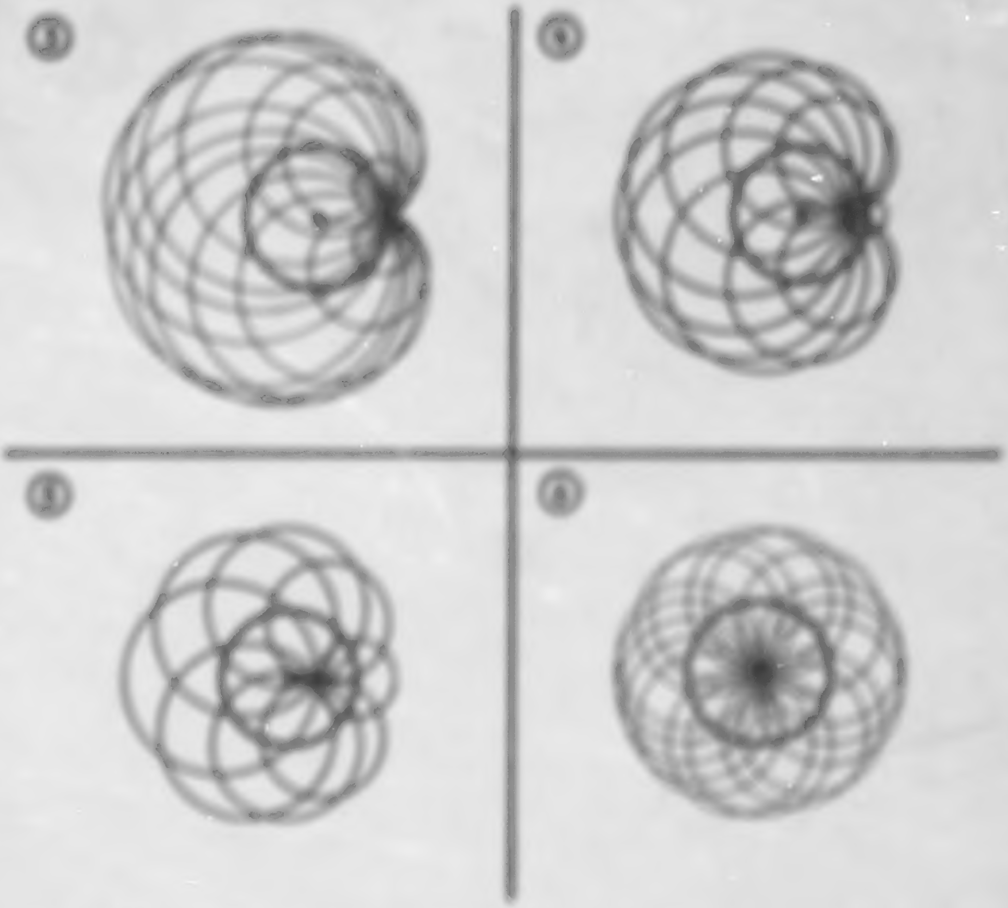
①



②



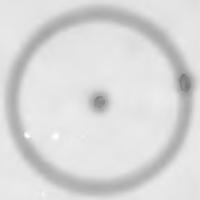
G 2



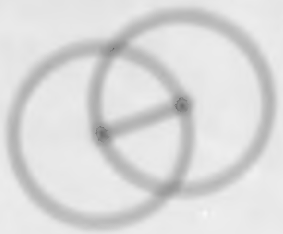
Exercise 2

The object of this exercise is to construct a lattice similar to those used for the Purchase Vector Space lessons in Part IV of the CSMP Intermediate Grades curriculum.

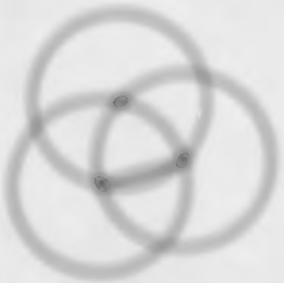
T: Here is another construction. I'll begin it at the board and state the rules. First make a dot in blue. Then draw a circle with the blue dot for center. Then mark another blue dot on the circle.



T: Draw another circle with the second blue dot for center, but do not change the radius of the compass.



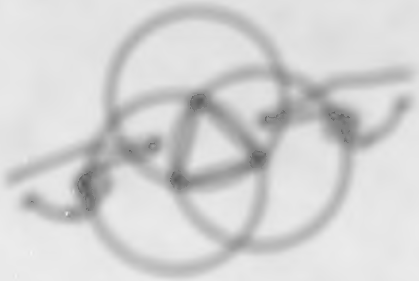
T: Use a straightedge to connect the blue dots with a red line segment. Then pick a point where the two circles meet. Draw a blue dot, and then draw another circle with that blue dot for center and with the same radius as the first two circles.



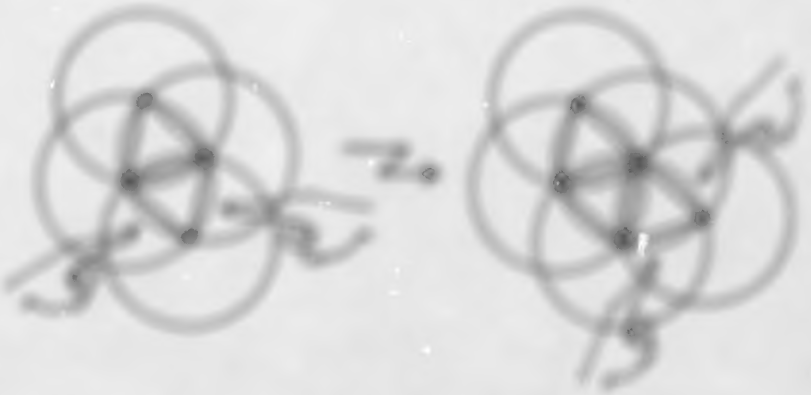
T: Draw red line segments from that center to the other blue dots on the circle.

G 2

The segments added at this stage are indicated in the drawing below.

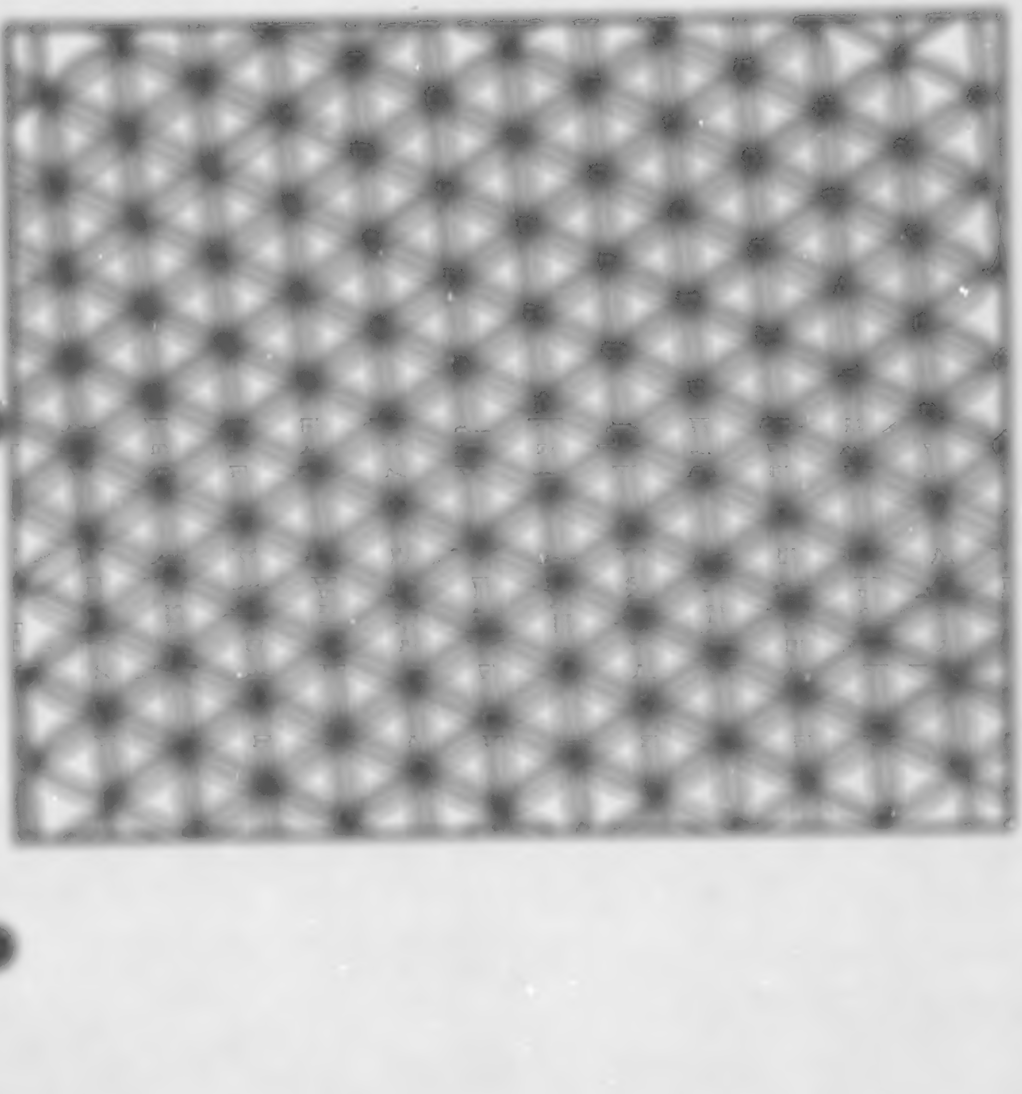


6. Continue by picking a point where two circles meet, drawing a blue dot, and drawing a new circle with that blue dot as center and with the same radius as the other circles. Then draw red line segments from the center to the blue dots on its circle.



7. Now start your own drawing and fill the whole page with circles.

A drawing of the type described is shown on the next page. Encourage accurate and careful construction by the students.



G 2

At the end of the lesson, allow a few minutes for discussing the drawings. Include these observations in your discussion:

- All of the red segments have the same length and that length is the radius of all the circles.
- There are three families of parallel lines.
- Each circle has at most six blue dots on it.
- The construction results in a drawing similar to those used to represent interactions in the [illegible] [illegible] [illegible] [illegible] of Part III. Some mention of [illegible] and [illegible] might avoid that situation in the class.

1.2 Perpendicularity [illegible]

SPECIFIC LEARNING OUTCOME:

Recognize the notion of a square corner and produce one by folding paper. Use a square corner to recognize line segments perpendicular to given line segments and containing given points. Recognize shapes with [illegible] square corners.

MATERIALS:

Teacher: Construction paper; demonstration [illegible]

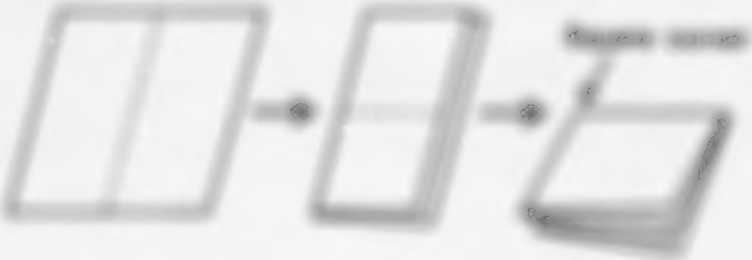
Student: Paper; pencil; Worksheets [illegible] and [illegible]

NOTE: [illegible]

DEVELOPMENT OF LESSON

Exercise 1

Begin by discussing square corners. Illustrate by providing one example, such as the corner of a piece of paper, and ask the students to provide other examples in the classroom. Discuss the problem of checking suggested examples. Demonstrate constructing a square corner by folding a piece of construction paper as illustrated here.



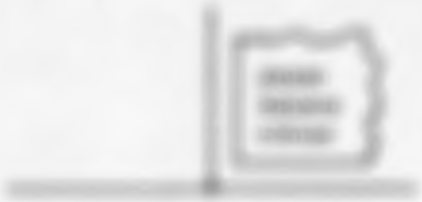
Instruct each student to produce his own [illegible] a square corner by folding a piece of paper. With each student equipped with a paper square corner, suggested occurrences of square corners in the classroom can be checked by individual students. Mention the special geometric shapes of rectangles and squares in the discussion of examples.

G 9

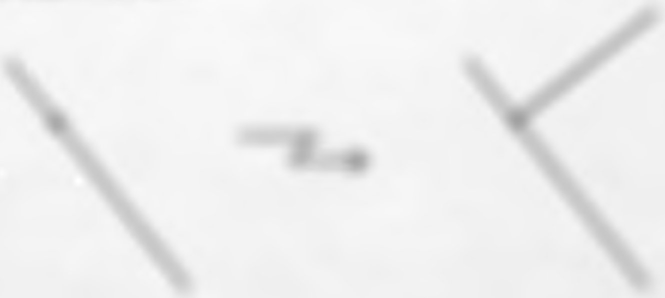
Draw a line segment and a dot on it on the board.

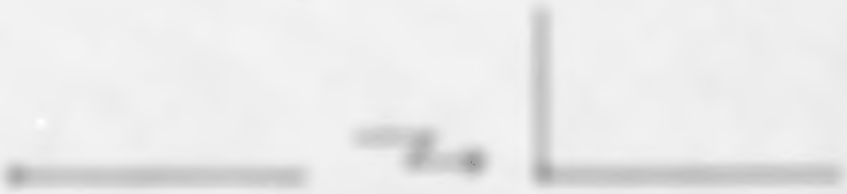
T: Let's make a square corner at this point with this line segment for one side of the corner. We can use a paper square corner as a test.

Ask a student to perform the construction using your paper square corner which, being made of construction paper, will receive important use. Several other students might check the construction with their paper square corners.

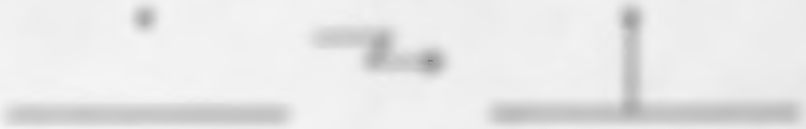


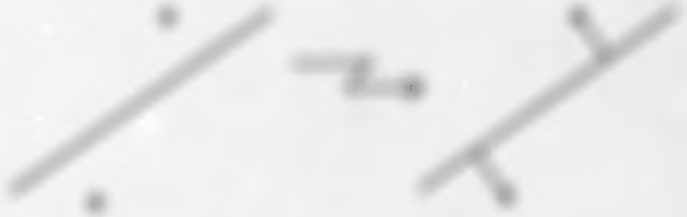
Similarly ask students to construct square corners in the following situations created by differing orientations of line segments and placements of dots. Suggest in each case that the line segments forming a square corner meet at the point of the indicated dot.





[illegible]





[illegible]

[illegible]

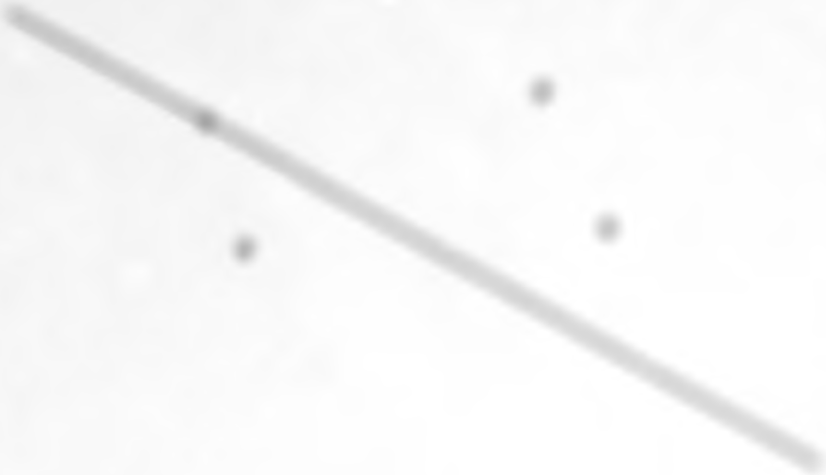
[illegible]

[illegible]

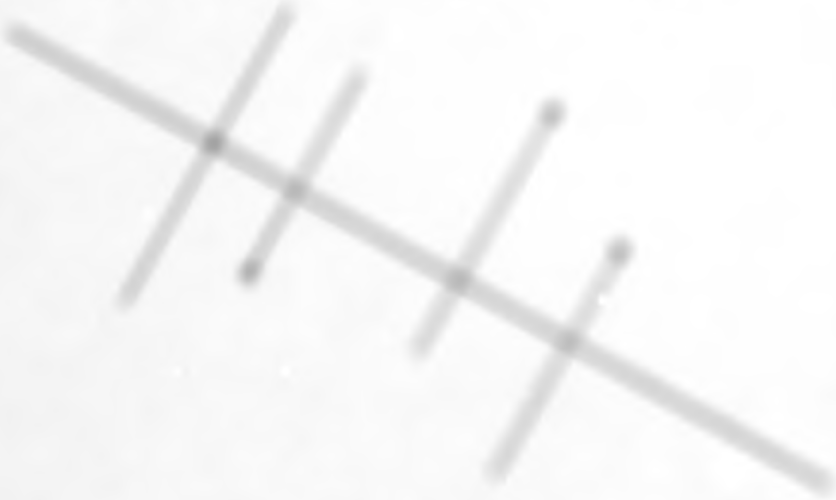
G 8

[illegible]

[illegible]



[illegible]



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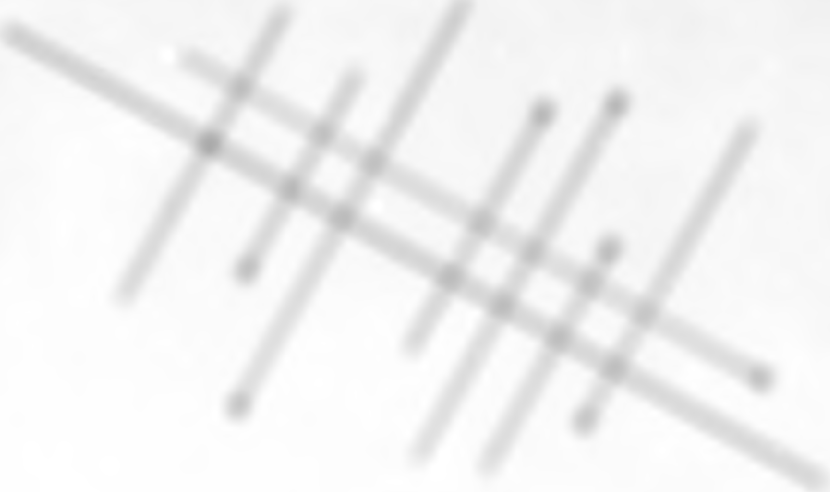

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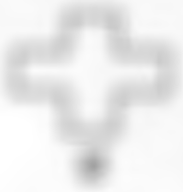
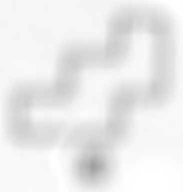
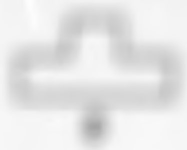
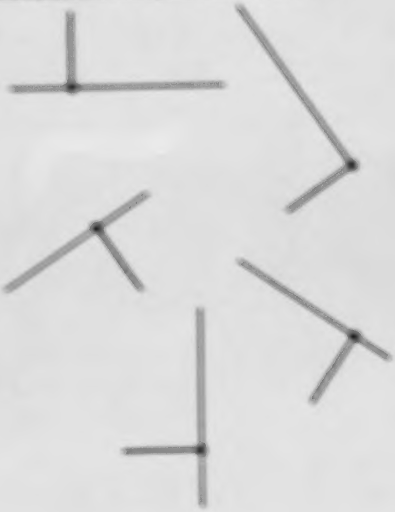






Name ____________________ C [illegible] *

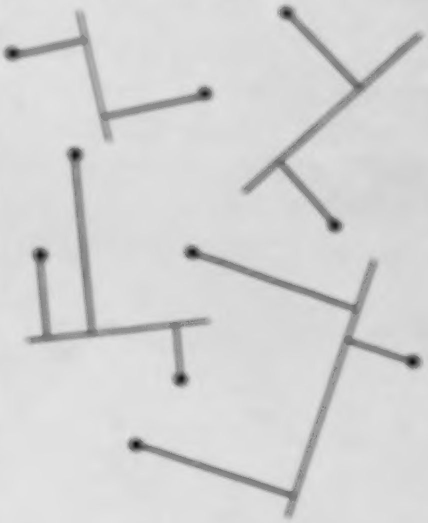
[illegible]



(Perpendiculars may be constructed on either side of the given line segments.)

Name ____________________ C [illegible] **

[illegible]



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1.4 Perpendicularity #2

CAPSULE LESSON SUMMARY

Use a compass to find the midpoint of a line segment and observe that the construction yields also a line segment perpendicular to the given line segment. Develop methods of using the compass to construct perpendicular lines to given lines and through given points.

MATERIALS

Teacher: Chalkboard compass; straightedge
Student: Paper; pencil; individual compass; straightedge; Worksheets G03, G04, and G05

DESCRIPTION OF LESSON

Exercise 1

Draw a line segment about 40 cm long on the board.

T: Here is a line segment. What is the midpoint of the line segment?

S: The point that is right in the middle.

S: The midpoint is the same distance from each end.

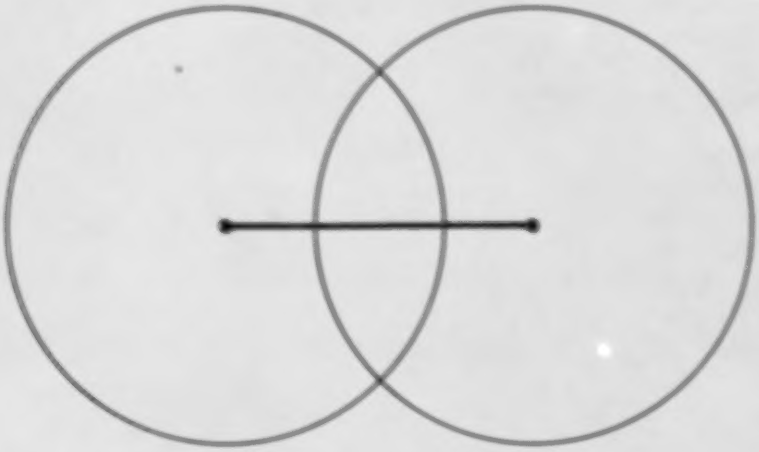
T: How can we find the midpoint?

S: Measure the line segment and take half.

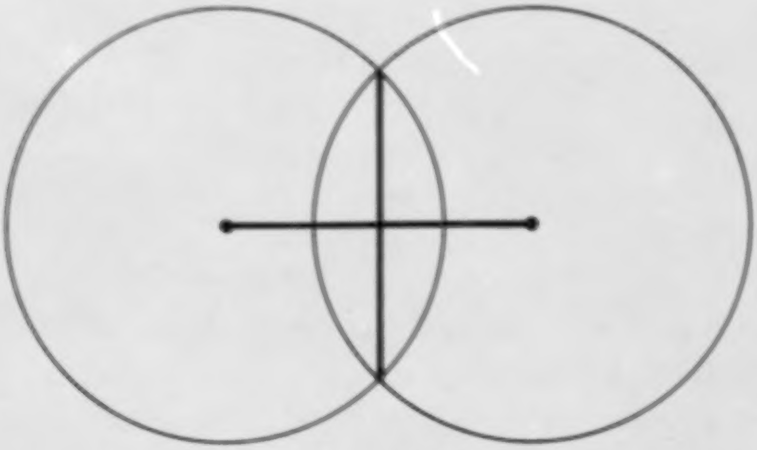
S: Divide it in two equal parts.

T: But today we will not use a ruler. The problem is to find the midpoint of the line segment with a compass. That is we must divide the line segment into two pieces of the same length using a compass. We may use a straightedge for drawing straight line segments, but we may not use a ruler for measuring.

Let students experiment at the board with the demonstration compass and encourage the class to discuss the attempts. If necessary, guide the discussion to suggest the construction of two circles of the same radius, each centered at an endpoint of the line segment.



Often students will spontaneously set the radius of the compass to the length of the line segment although this is not necessary. However, if the radius is so small that the circles do not intersect, a longer radius will be needed. Having constructed the two circles, the direction of the discussion should be to observe that the line connecting the two intersection points crosses the original line segment at its midpoint.



T: How can we check that we have found the midpoint?

S: Measure the two pieces.

T: We have no ruler.

S: Use the compass. Set its radius to the length of one piece and compare it to the other.

T: Let's do it.

Next, point to one of the intersection points of the circles with one hand and indicate the endpoints of the line segment in turn with the other.

T: Is this point (the intersection point) farther away from the first endpoint or the second endpoint?

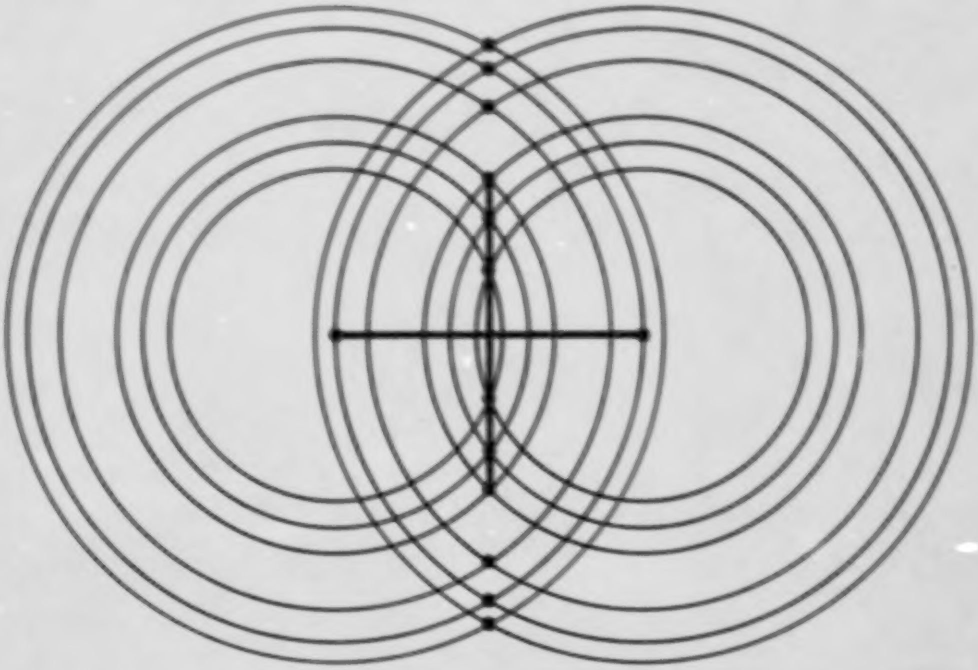
S: The distance is the same, because for each endpoint the distance is the radius of the circle centered at the endpoint, and the circles have the same radius.

T: Could we use a different radius for the circles and still have a construction that works for us?

S: The radius of the circles must be more than half the length of the original line segment.

T: Let's check with another radius.

Do so with several radii. Each time mark the intersections with red dots.



T: What else do you observe about the two line segments? (Trace them with a finger.)

S: They meet in a square corner.

S: So they are perpendicular.

T: Let's check that with my paper square corner.

Do so. Then provide Worksheet G03.

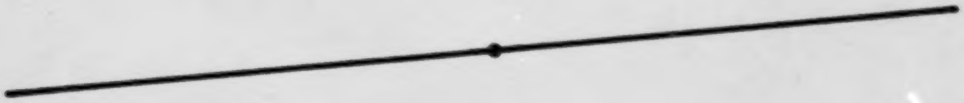
T: On this worksheet, use your compass and straightedge to find the midpoint of each line segment. Do not use a ruler for measuring.

Allow several minutes for individual work helping those students who have difficulty. After most students have completed at least two constructions, continue with Exercise 2.

Exercise 2

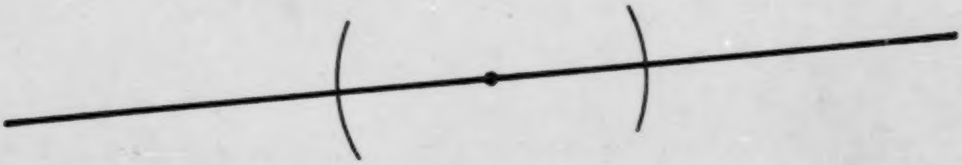
T: The compass construction of midpoints gives us a bonus; we find square corners as well. The line we construct is perpendicular to the given line segment. Now suppose that we are given a line segment and want to use the compass to find a line segment perpendicular to it.

Draw a line segment and mark a point on it. Make it very long to remove its endpoints from the construction and to give the impression of a line instead of a line segment.

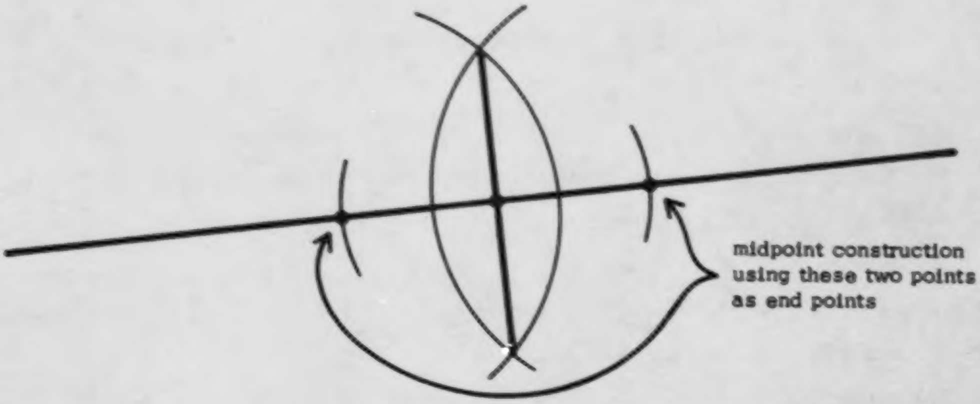


T: We know how to draw a perpendicular to this line at the indicated point using a paper square corner, but how can we do so using only a compass and a straightedge?

Lead the discussion to invert the midpoint construction, that is, to create a line segment along the given line for which the given point is a midpoint. Invite a student to use the compass to locate the endpoints of such a segment.



Then the midpoint construction will yield the desired perpendicular segment.

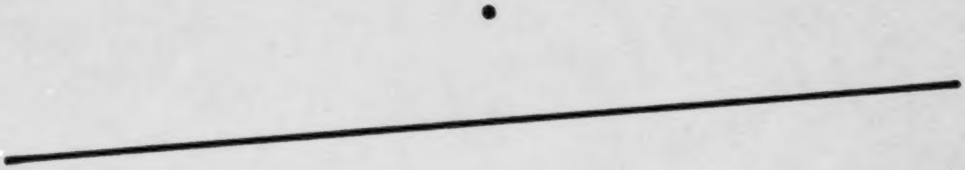


If less than five minutes remain, proceed with Worksheet G04. Otherwise, include the following exercise before the worksheets.

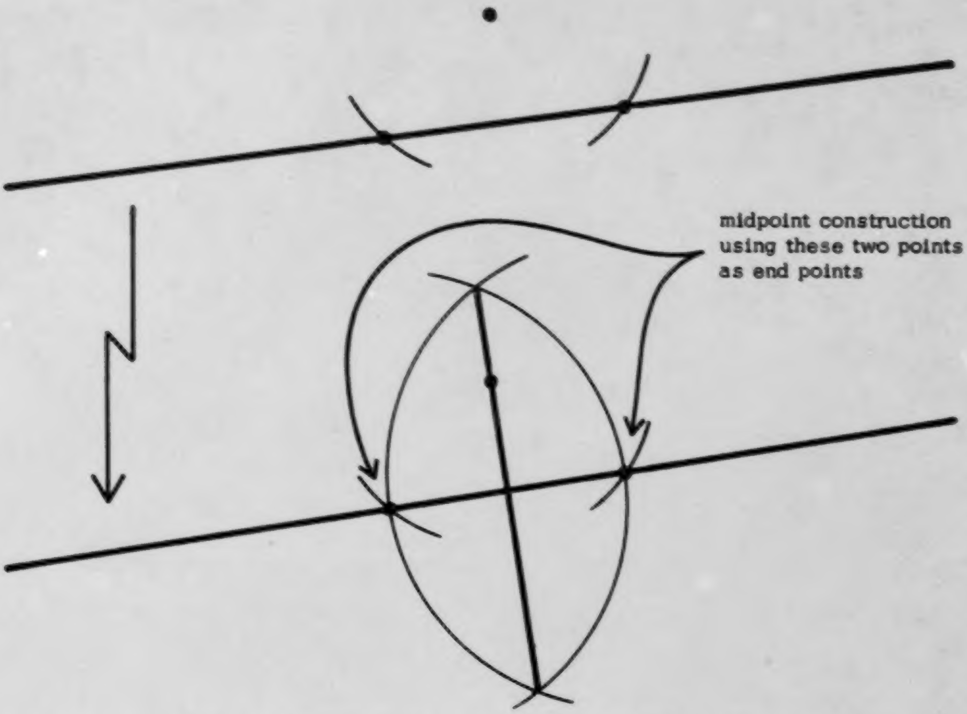
Exercise 3

T: Here is another problem which we solved using paper square corners. First we draw a line segment and a point. The problem is to draw a line segment through the given point which is perpendicular to the given line.

Again draw a long line segment across the board and mark a point not on the line.

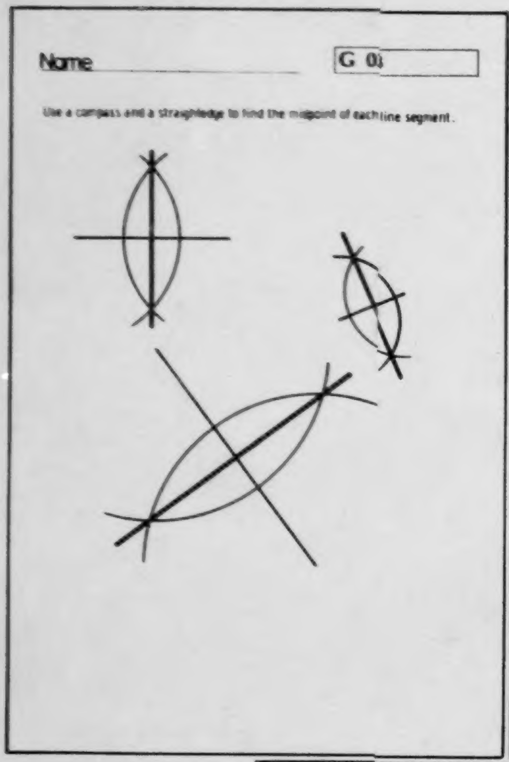


The discussion should aim to use the midpoint construction again. The idea is to use the fact that in the midpoint construction, each of the intersection points of the circles was the same distance from the endpoints of the given segment. The construction is similar to that of Exercise 2. Use the compass to locate two points on the line which are equidistant from the given point. These points will define a line segment with which to perform the midpoint construction.



midpoint construction using these two points as end points

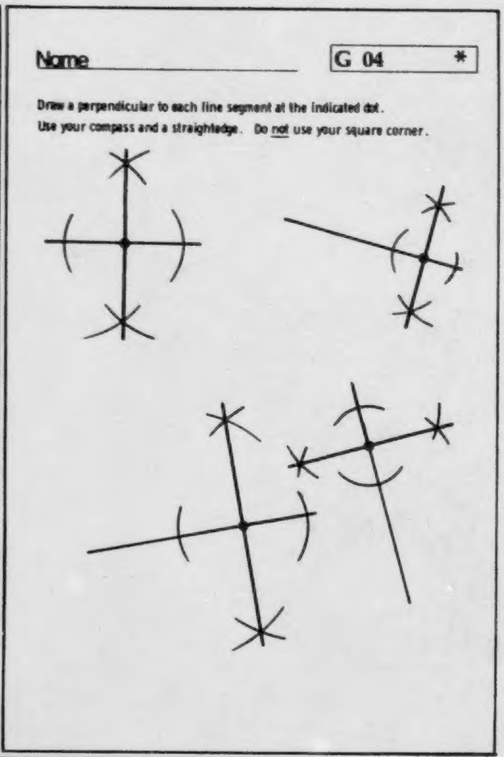
At the end, provide Worksheets G04 and G05 for individual practice of the constructions. Encourage students to construct line segments through the centers of the indicated dots, and when necessary, to place the point of the compass at the center of a dot.



Name

G 03

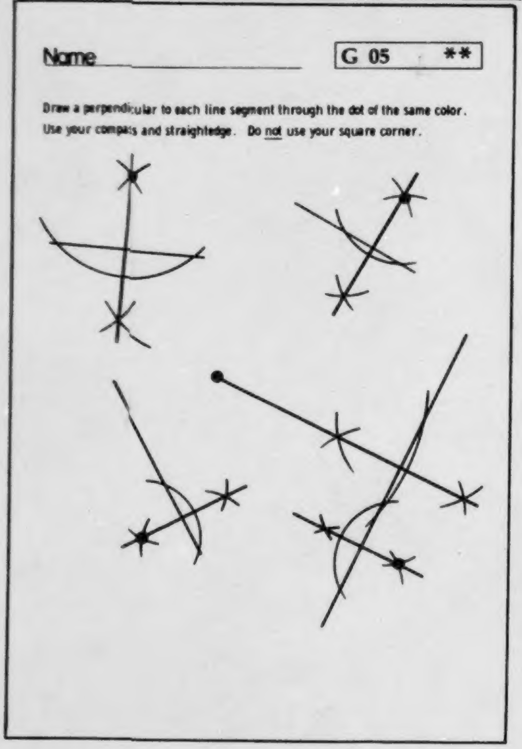
Use a compass and a straightedge to find the midpoint of each line segment.



Name

G 04 *

Draw a perpendicular to each line segment at the indicated dot.
Use your compass and a straightedge. Do not use your square corner.



Name

G 05 **

Draw a perpendicular to each line segment through the dot of the same color.
Use your compass and straightedge. Do not use your square corner.

1.5 Polygons #1

CAPSULE LESSON SUMMARY

Use a compass and a straightedge to duplicate line segments. Discuss a variety of polygons and then use a compass and a straightedge to construct equilateral, eight-sided polygons.

MATERIALS

Teacher: Chalkboard compass; straightedge

Student: Paper; pencil; individual compass; straightedge

DESCRIPTION OF LESSON

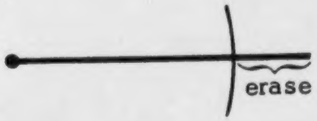
Exercise 1

Draw a line segment about 40 cm long on the board.

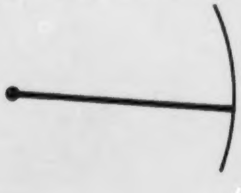
T: I have here a compass and a straightedge. The problem is to construct another line segment exactly the same length as this one.

Invite students to experiment with constructions. Someone may suggest making a mark on the straightedge. Respond with a prohibition and a remark against making a ruler. Guide the discussion to one of two constructions.

- Draw a second long line segment. Set the compass radius to the length of the original segment and then place the compass point at one endpoint of the second segment. Draw a portion of a circle intersecting the segment. Erase the excess.



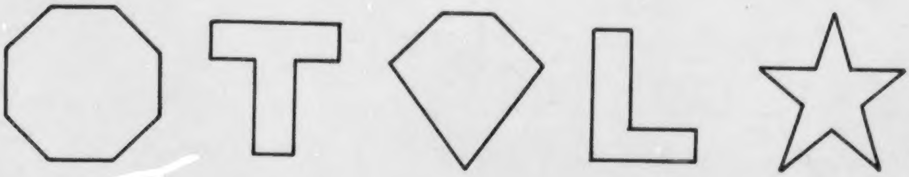
- Set the compass to the length of the given segment. Draw a dot, set the compass point at the dot, and draw a portion of a circle. Use the straightedge to draw a segment from the dot to any point on the circle.



G 5

Exercise 2

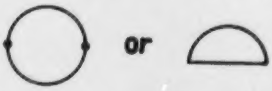
Lead a short discussion of shapes. Focus attention on shapes with sides which are line segments. In the discussion, familiar names should be mentioned, such as, "square", "rectangle", "triangle", and "parallelogram". Other names might be introduced in passing, such as, "pentagon", "hexagon", "octagon", and the like. Ask students for common examples of each shape. They might mention a homeplate (from baseball), a honeycomb cell, and a stop sign. Elicit the observation that the sides might be, but need not all be, of the same length and ask students to sketch examples on the board.



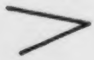
Use the discussion to enunciate the rules for shapes under discussion:

- line segments for sides;
- closed;
- no dangling pieces;
- each side meets only two other sides.

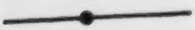
Lead students to realize that any number of sides is possible, in principle, except two or one. In our development class, several solutions for two sided shapes were offered and rejected by the class.



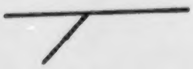
no, may only use line segments



no, not closed



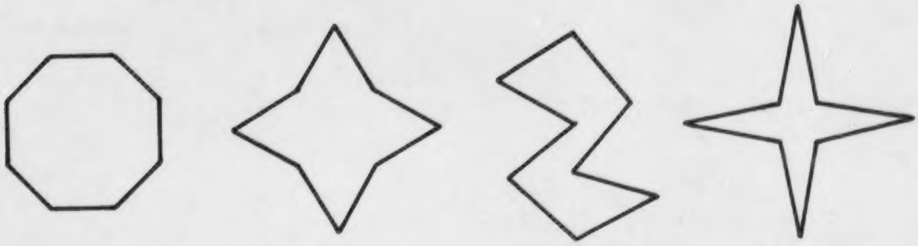
(drawn in two strokes) no, not closed



no, dangling pieces not allowed

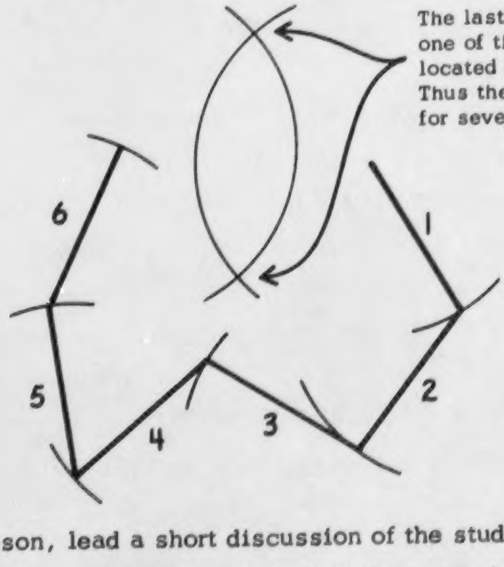
T: Here is your problem. Use a compass and a straightedge to draw a shape with eight sides such that all sides have the same length.

Monitor the individual work. As students produce solutions, ask for another with a smaller or larger area, but using the same length side. Some solutions are shown here.



If any student has more time before the final collective discussion, ask for shapes with an odd number of sides.

Note that the key to the construction lies in bringing the ends of the chain of sides close together when the sixth side is added. Then the location of the seventh and eighth sides are forced to close the shape. There are usually two choices for the location of the last corner and these are located by finding the intersection of the circles centered on the free endpoints as shown here.



The last corner must be one of these two points, located with a compass. Thus there are two choices for seventh and eighth sides.

At the end of the lesson, lead a short discussion of the students' experience with the problem focusing on the technique of closing the shape.

G 6

1.6 Polygons #2

CAPSULE LESSON SUMMARY

Use a compass and a straightedge to construct an equilateral, six-sided polygon; an equilateral, four-sided polygon; and a regular (equilateral) triangle. Given two line segments, use a compass and a straightedge to construct as many essentially different four-sided shapes as possible with each side the same length as one of the given line segments. Make some observations concerning the variety of shapes that can be constructed and the possible diagonal length of certain of the shapes.

MATERIALS

Teacher:	Chalkboard compass; straightedge; colored chalk
Student:	Paper; pencil; individual compass; straightedge; Worksheet G06

DESCRIPTION OF LESSON

Exercise 1

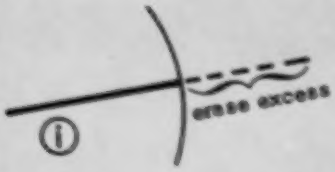
Draw a line segment about 40 cm long on the board.

T: Let's draw a six-sided shape whose sides are each as long as this line segment. We will use only a compass and a straightedge.

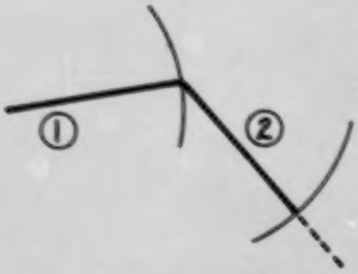
Invite students in turn to add a side to the drawing and encourage class discussion of the process. Enforce the rules of Lesson G1.5 as the drawing progresses. Some students may want just to guess at locating corners of the shape being drawn and may need prompting from their peers on the use of the compass to assist in the construction. At each step ask for some indication of any restriction on the availability of choices. For example, the first side can be drawn anywhere. The second side need only be connected to the first and the third side to the second. There are infinitely many choices in each case. But the fourth side cannot he placed arbitrarily if the shape is to be closed.

G 6

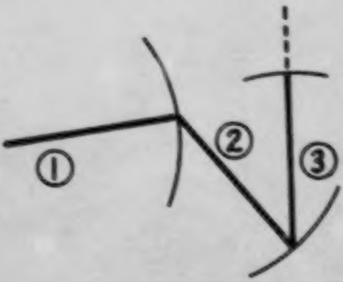
① can be placed anywhere.

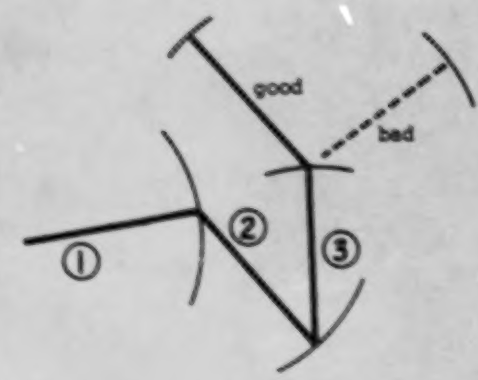


② must join ①.

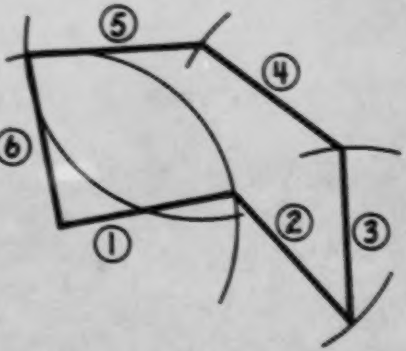


③ must join ②.



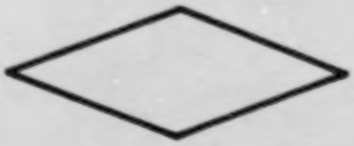


④ must be carefully located. The free end of ④ must be near enough to the free end of 1 to be joined in two more steps. Two choices are are shown, one good and one bad.



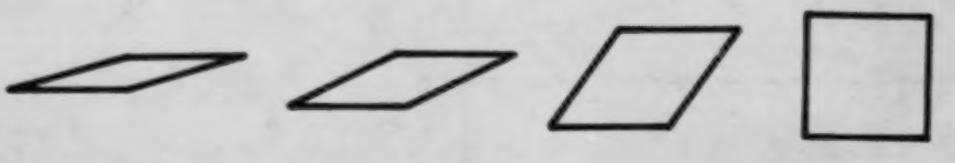
⑤ and ⑥ are placed by swinging arcs from the two free ends and picking an intersection point for the last corner. There might be two choices, but in this illustration only one is viable.

Repeat the exercise by drawing a four-sided shape using the same length for each side. For example,



T: This shape is called a rhombus. We could draw many different rhombi with the same length side.

Sketch a few examples to illustrate the progression shown here.



T: One of them has a special name.

S: The square.

T: Right; a square is a rhombus with square corners. Now let's draw a three-sided shape with each side this length.

S: You mean a triangle.

T: Yes, a triangle.

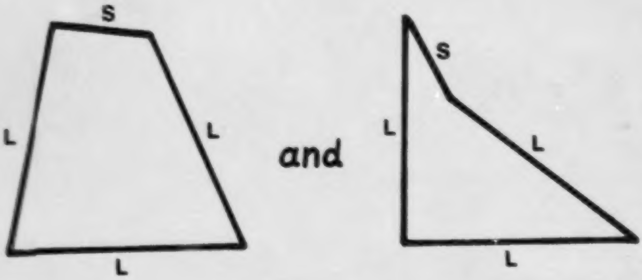
In the process, the class should realize that while many rhombi (all sides the same length) can be drawn with a given length side, only one equilateral triangle can be drawn with a given length side. Emphasize this fact.

Exercise 2

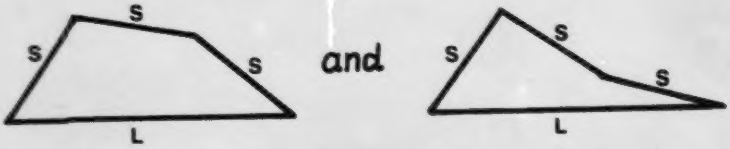
Distribute Worksheet G06.

T: On this worksheet there are two line segments, A and B. Use your compass and straightedge to draw as many different four-sided shapes as you can. Each side of a shape must be the same length as one of these two segments.

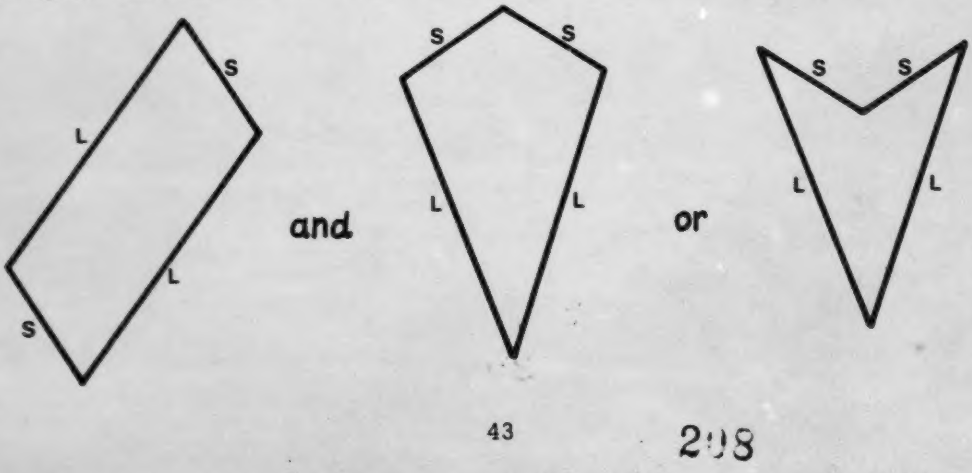
There are five combinations of sides : all short; all long; one short and three long; three short and one long; and two short and two long. In the first two cases, the resulting shape is a rhombus as already discovered and discussed. There are many different-sized rhombi in each case. In the case of one short and three long, there is again essentially only one solution (with respect to the arrangement of sides); that is, short-long-long-long, even though the area may vary. For example,



The case of three short and one long is similar; there is essentially one solution, short-short-short-long. Again the area may vary, for example

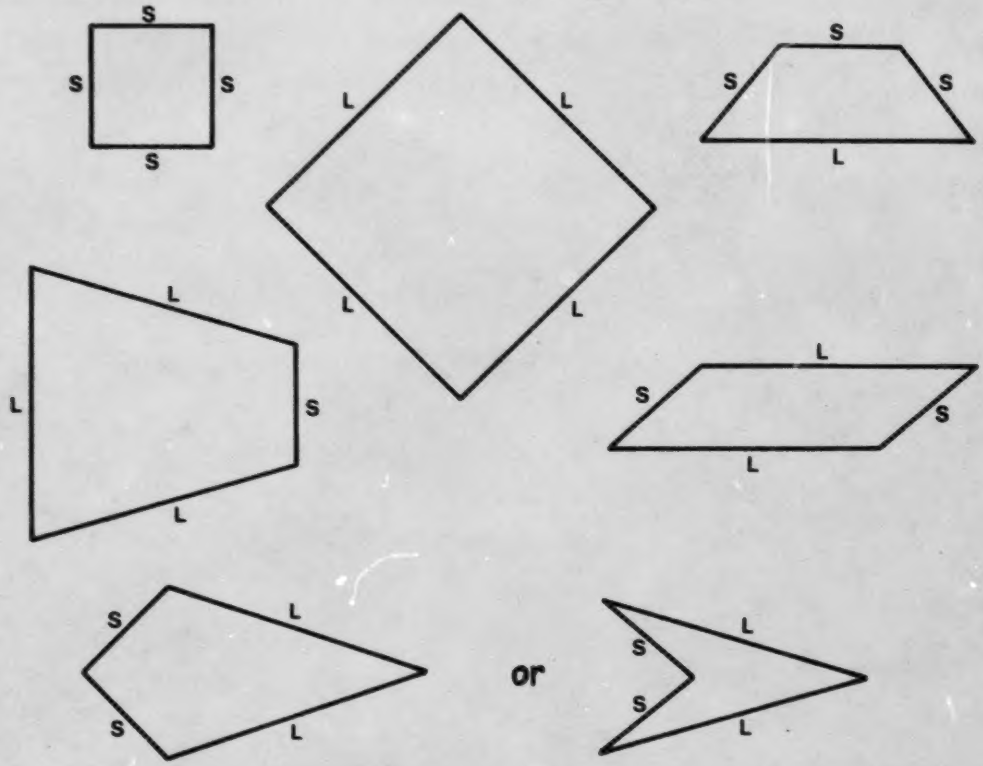


The case of two short and two long is more interesting since there are essentially two solutions, depending on the order of the sides in rotation: short-long-short-long or short-short-long-long. The first sequence results in a parallelogram and the second in a kite or wedge.



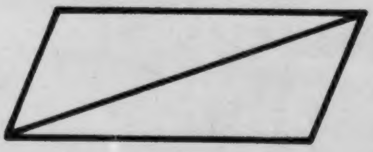
G 6

Encourage the students to find examples of each type rather than to repeat examples from the same case. A full set is shown here.



Near the end of the lesson, collect examples on the board so that students can compare their collections and decide which are of the same type. Also, let students spend some time comparing drawings among themselves to note the different sizes possible within each type.

For the final discussion, focus on the parallelogram. Draw the longer diagonal in red.



T: What happens to this red line segment as you change the size of the parallelogram?

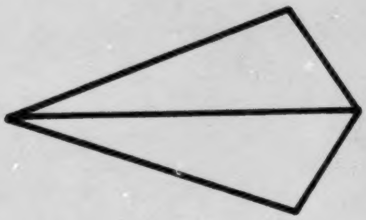
S: It gets longer if you squeeze the parallelogram together.

T: How long can it be?

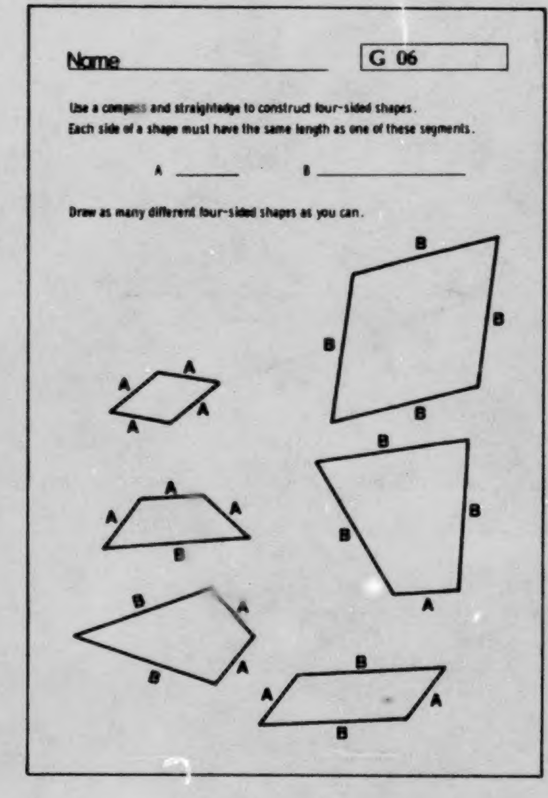
S: No longer than the sum of the lengths of the short and the long sides.

T: Now look at this kite.

Draw the longer diagonal in red.



S: The same is true. The red segment cannot be longer than the sum of the lengths of a short side and a long side.



Name ____________________ G 06

Use a compass and straightedge to construct four-sided shapes.
Each side of a shape must have the same length as one of these segments.

A ________ B ____________________

Draw as many different four-sided shapes as you can.

A
A
A
A
B
B
B
B
A
A
A
B
B
B
B
A
B
A
A
B
B
A
A
B

1.7 Triangle Constructions

CAPSULE LESSON SUMMARY

Given two line segments, construct as many essentially different triangles as possible with each side the same length as one of the line segments. Discover a rule relating the length of the segments to the number of different triangles. Repeat this activity with three given line segments and discover the triangle inequality. Pose the problem of breaking a stick into three pieces and deciding whether a triangle can be formed with the three pieces.

MATERIALS

Teacher:	Chalkboard compass; straightedge
Student:	Pencil; individual compass; straightedge; Worksheets G07 and G08; paper

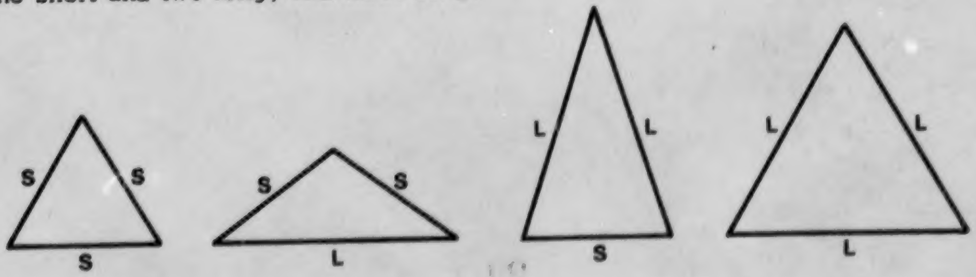
DESCRIPTION OF LESSON

Exercise 1

Draw two line segments on the board, one about 25 cm long and the other about 40 cm long.

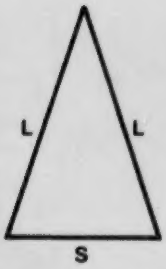
T: Here I have two line segments. Let's draw a triangle using a compass and a straightedge. Each side of the triangle is to be the same length as one of these line segments.

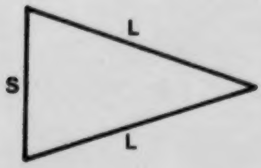
Invite students to perform the construction on the board with class discussion of the technique. There are four essentially different triangles depending on the number of each length side chosen: three short; two short and one long; one short and two long; and three long.

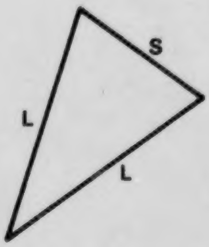


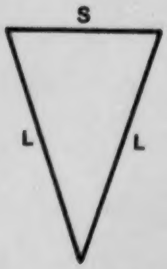
G 8

If an equilateral triangle is constructed either with three short or with three long sides, comment on its correctness and then introduce the restriction that each of the two lengths must be used at least once in each triangle. During the discussion introduce the idea that the orientation of the triangle on the board does not alter it in any way important to its identity with respect to the choice of sides. Thus, these four triangles should all be thought of as essentially the same.









With the class secure in the triangle construction technique and with the two triangles (short-short-long and long-long-short) constructed on the board, provide Worksheet G07 where there are five pairs of line segments. Each student is to construct as many triangles as possible with each pair of segments. Each side of the triangle must be as long as one member of the pair and each member of the related pair of segments must be used at least once. The pairs are denoted "A", "B", "C", "D", and "E". Tell the students to mark each triangle with the same letter as the pair of segments used to construct it. As students fill up the space on the worksheet, provide them with unlined paper.

Encourage accurate and careful construction. Allow time for experimentation and for the conviction to grow that in no case are there more than two triangles possible. Students will discover these cases. Encourage them to try to formulate a rule to predict the possibility of one or two triangles and then to draw pairs of segments to test their rules. Provide rulers to those who request them for further testing of a rule.

Collect the results in a table on the board.

	number of triangles	short segment	long segment
A	2	3cm	5cm
B	1	4cm	9cm
C	2	5cm	7cm
D	1	2cm	4cm
E	1	3cm	7cm

Lead the discussion to elicit the rule for finding two triangles. There are at least two good ways to state the rule:

- The short segment must be more than half as long as the long segment,

or

- Twice the length of the short segment must be more than the length of the long segment.

Develop class understanding of the rule by prescribing several pairs of lengths and asking for the number of triangles.

short	long	number of triangles
3cm	5cm	2
15cm	25cm	2
15cm	30cm	1
15cm	40cm	1
1007cm	2003cm	2
4.5cm	8cm	2

Exercise 2

T: Now I will give you three segments with which to draw triangles. Each must be used once in each triangle.

Provide Worksheet G08 for individual work. Proceed as with the first worksheet. After the individual work, collect the results on the board.

	lengths			number of triangles
A	3cm	4cm	5cm	1
B	2cm	4cm	7cm	0
C	1.5cm	3.5cm	4cm	1
D	2.5cm	3.5cm	6cm	0

Discuss the results with the purpose of formulating a rule for the possibility of constructing a triangle. A good formulation is:

- the sum of the lengths of shorter segments must be more than the length of the longest segment.

Exercise the students' application of the rule by providing several sets of lengths and asking whether or not a triangle can be formed.

lengths			triangle?
1cm	6cm	8cm	No
2cm	6cm	8cm	No
3cm	6cm	8cm	Yes
107cm	412cm	500cm	Yes
4.3cm	9.8cm	15cm	No

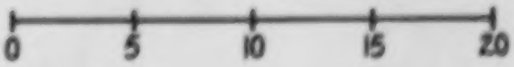
Exercise 3

Devote the last few minutes of the lesson to introducing the problem to be solved in the next four lessons. The problem provides a context in which to study the topic of building a triangle from given line segments.

T: I have a stick which is 20 cm long. Suppose that I break the stick in two places to make three pieces. Will I be able to make a triangle with the pieces?

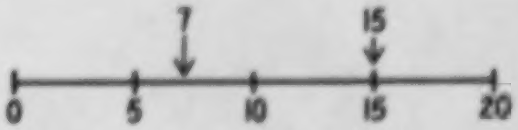
S: Sometimes you will, but only if the two smaller pieces are together longer than the largest piece.

Draw a line segment and label it as below to represent your stick.



T: What are some possibilities?

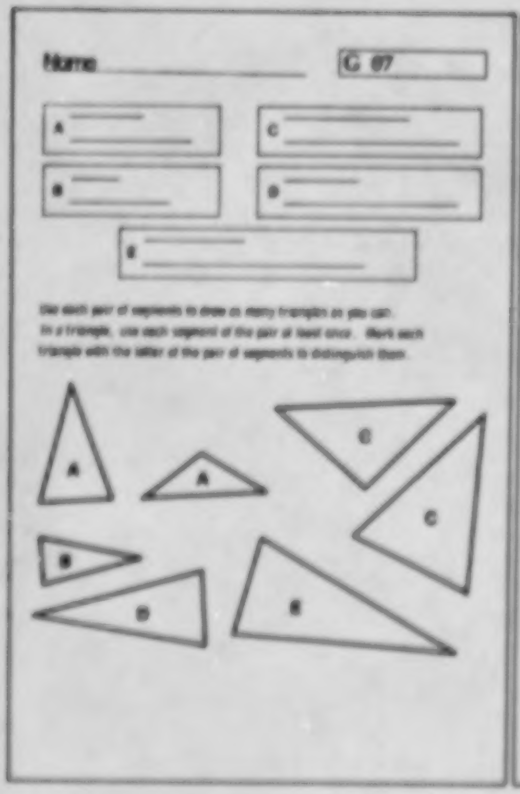
S: You could break it at 7 and at 15. I think that you could make a triangle then.



T: How long would the three pieces be?

S: 7 cm, 8 cm, and 5 cm; 7 + 5 = 12 and 12 > 8. That works!

Ask for several more cases, some that work and some that do not.



Name

G 07

A

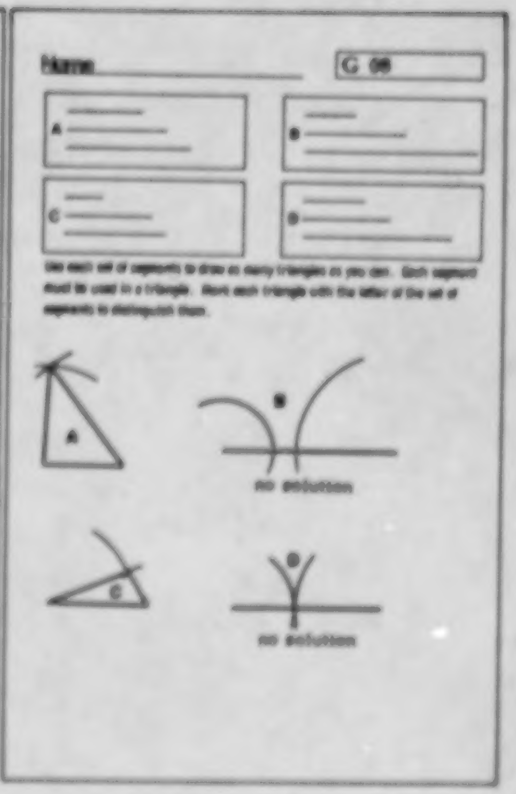
C

B

D

E

Use each pair of segments to draw as many triangles as you can. In a triangle, use each segment of the pair at least once. Mark each triangle with the letter of the pair of segments to distinguish them.



Name

G 08

A

B

C

D

Use each set of segments to draw as many triangles as you can. Each segment must be used in a triangle. Mark each triangle with the letter of the set of segments to distinguish them.

A
B
no solution
C
D
no solution

217

1.8 Breaking a Stick #1

CAPSULE LESSON SUMMARY

Introduce the problem of breaking a stick into three pieces and trying to form a triangle. Examine many cases to decide the conditions for success and for failure. Rephrase the triangle inequality.

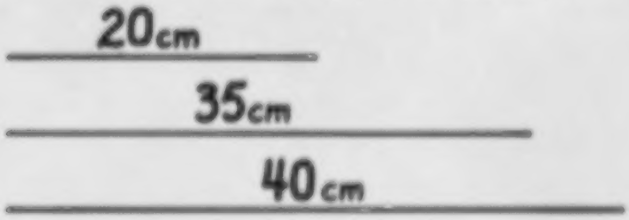
MATERIALS

Teacher: Colored chalk; straightedge

Student: Colored pencils; Worksheets G09 and G10

DESCRIPTION OF LESSON

Draw three line segments of the indicated lengths on the board.



T: Could you make a triangle using each of these line segments exactly once?

S: Yes, 20 + 35 = 55 and 55 > 40.

T: What is the rule?

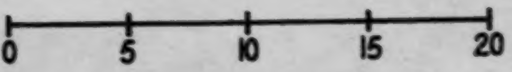
S: When the two shorter segments are put together the total length must be greater than that of the longest segment.

Give a few more sets of lengths while building a table.

lengths	triangle?
20cm 35cm 40cm	Yes
20cm 35cm 70cm	No
20cm 35cm 150cm	No
35cm 20cm 150cm	No
567cm 20cm 35cm	No
3cm 4cm 5cm	Yes

G 9

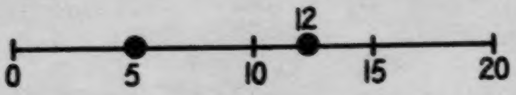
Draw this picture of a stick on the board.



T: Let's think about a stick 20 cm long. Breaking in two places makes three pieces. My question is: what is the probability of being able to make a triangle with the pieces, if the breaking points are chosen at random?

Allow the students to discuss this question.

T: Let's use blue dots to mark the breaking points in the picture. Suppose that we put a blue dot at 5 and a blue dot at 12. Is this a success; can we make a triangle?



S: We need to find the lengths of the pieces.

Begin a table.

breaking points		length	triangle?
●	●		
5	12	5cm 7cm 8cm	Yes

Now lead the discussion to continue the table with more choices for the blue dots. Allow students to make the choices. With each choice, ask students to determine the lengths of the pieces and to apply the rule to determine the success or failure of the choice. After a few trials, begin to ask specifically for a success or for a failure. Add some variety by asking one student to specify one blue dot (breaking point) and call for a successful choice for the other blue dot from another student. Be sure to include one or two cases in which the length of one piece is 10 cm and hence the sum of the two shorter lengths is exactly 10 cm. These cases are failures since 10 cm is not greater than 10 cm. Your table should resemble the following.

breaking point		lengths			triangle?
●	●				
5	12	5cm	7cm	8cm	Yes
6	11	6cm	5cm	9cm	Yes
3	13	3cm	7cm	10cm	No
1	4	1cm	3cm	16cm	No
18	7	2cm	7cm	11cm	No
$6\frac{2}{3}$	$13\frac{1}{3}$	$6\frac{2}{3}$cm	$6\frac{2}{3}$cm	$6\frac{2}{3}$cm	Yes
1.4	14.2	1.4cm	12.8cm	5.8cm	No

When a large number of entries have been added to the table, lead another discussion of the rule. Ask if anyone can state another rule with the same effect, one that tells how large the pieces may be for success. The object is the equivalent rule:

- Each piece must be shorter than half the length of the stick (in this case 10 cm).

To elicit this rule, direct attention to the list of lengths in the table. Successes occur exactly when all lengths are less than 10 cm. In the discussion someone might comment that if the longest length is more than 10 cm, then the other two pieces together must be shorter than 10 cm, since only 20 cm are available in the whole stick. So with the largest piece longer than 10 cm, there is a failure.

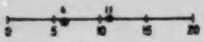
An articulate discussion of the equivalence of the two rules is not necessary, but some conviction on the part of the students is necessary since this is the rule which will be used in the sequel.

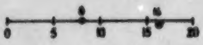
After discussion of this rule, proceed with individual work on Worksheets G09 and G10 to provide practice with the rule in determining successful choices of points. As students begin working on G10, explain that the red dots are used only to indicate that certain points cannot be breaking points, whereas every point between the red dots could be a second breaking point.

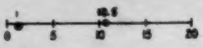
Name ____________________

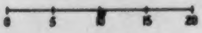
G 09 *

In each picture, one breaking point is given. Find, if possible, a second breaking point that gives a triangle.

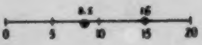

6
11
0
5
10
15
20

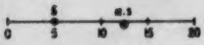

8
16
0
5
10
15
20


1
10.5
0
5
10
15
20



no solution


8.5
15
0
5
10
15
20

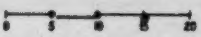

5
12.5
0
5
10
15
20

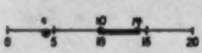
(Many solutions are possible.)

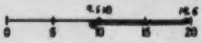
Name ____________________

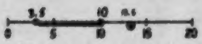
G 10 **

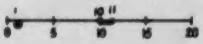
In each picture, one breaking point is given. Color blue all possible second points that give a triangle. If necessary, use red dots to indicate points which cannot be a second breaking point. An example is done for you.

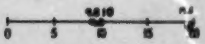



4
10
14
0
5
10
15
20


9.5 10
19.5
0
5
10
15
20


3.5
10
13.5
0
5
10
15
20


1
10 11
0
5
10
15
20


9.5 10
19.5
0
5
10
15
20

1.9 Breaking a Stick #2

CAPSULE LESSON SUMMARY

Review the problem of breaking a stick introduced in Lesson G1.8 and decide that a method of choosing two breaking points at random must be found. Using parallel projections (with a translator), establish a means for making this random choice.

MATERIALS

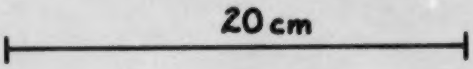
Teacher: Colored chalk; demonstration translator
Student: Colored pencils; individual translator;
Worksheets G11 and G12

NOTE: Your classroom set of materials might not include a demonstration translator or individual translators in which case borrow these materials from a CSMP fourth or fifth grade class. See the NOTE ON TRANSLATORS in the introduction to this chapter.

DESCRIPTION OF LESSON

Spend a few minutes at the beginning of the lesson reviewing the technique of using the translator.

Draw a line segment about 1 m long on the board and indicate that you will imagine it to be a 20 cm stick.



T: Who will tell us the problem of breaking the stick?

S: You have a stick 20 cm long and want to break it in two places to make three pieces. When can you form a triangle with the pieces?

T: And what is the rule?

S: The two shorter pieces together must be longer than the longest piece.

S: Each piece must be shorter than 10 cm.

Write the second statement of the rule on the board for emphasis.

T: But there is another part to the problem involving a question about probability.

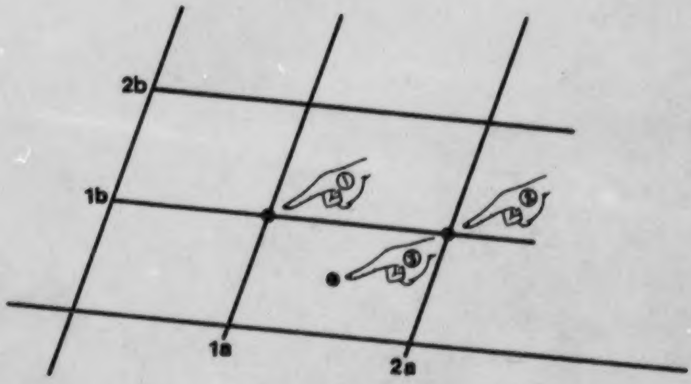
S: You choose the breaking points at random and want to find the probability of being successful at making a triangle.

T: Yes, so we must decide how to choose two breaking points at random.

Lead a short discussion of students' suggestions of random devices (spinners, darts, or whatever) for choosing the breaking points.

T: Last year we studied a situation in which two points on special lines were related to a single point. Remember the fruit market that offered only apples and blueberries? We studied transactions at this market. We drew a grid to represent the transactions.

Quickly draw a grid with a translator as follows.



T: Each point in the grid corresponds to two points, one on the apple line and one on the blueberry line. To which points does this point (indicate ①: 1a + 1b) correspond?

S: 1a on the apple line and 1b on the blueberry line.

T (indicating ②: 2a + 1b): And this one?

S: 2a on the apple line and 1b on the blueberry line.

T (indicating ③: 1.5a + 0.5b): And this one? How do we find points on the apple line and on the blueberry line?

Invite a student to trace with a finger on the grid to show a technique.

T: We project onto the apple line parallel to the blueberry line and we project onto the blueberry line parallel to the apple line.

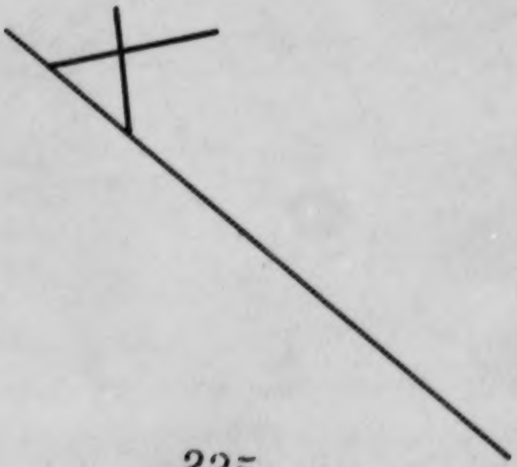
Use the translator to demonstrate the projections and ask again for the corresponding points.

S: 1.5a on the apple line and 0.5b on the blueberry line.

T: For our problem we need to choose two points on the same line, so we will use a slightly different trick, but we will still use the translator for parallel projections.

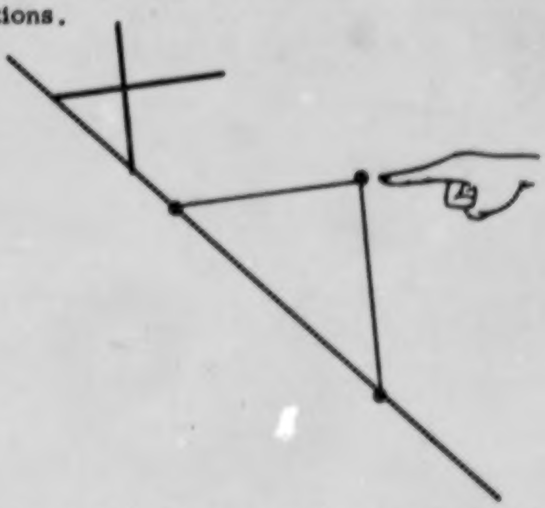
Erase the board except for the statement of the rule.

Draw a long line across the board at about a 45° angle. Then add a red segment and a blue segment as shown here.

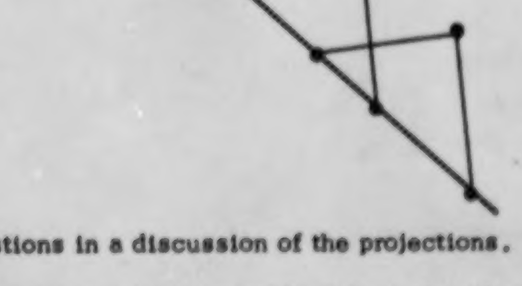


T: The red and the blue line segments give us two directions for projecting onto this line.

Draw a dot above the (first) line as indicated below. Use the translator to perform parallel projections as follows: align the translator with the red segment, roll to the dot, and draw a projection line. Then align the translator with the blue segment, roll to the dot, and draw a projection line. In each case, draw a dot of the appropriate color to indicate the image of the dot under the respective projections.



Repeat the double projection with another dot, this time with student assistance.



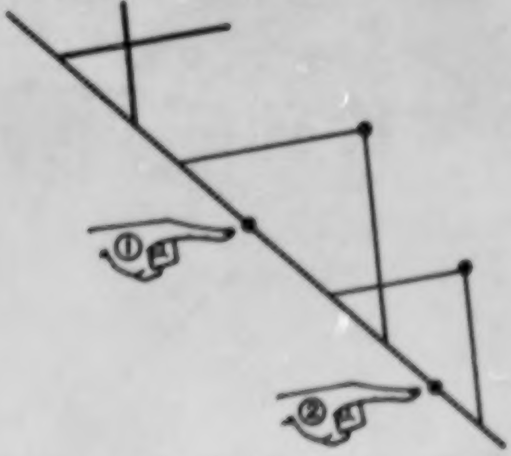
Raise these questions in a discussion of the projections.

- Does it matter whether we perform the red or the blue projection first? (No)

- How many image dots are produced from each chosen dot? (Two; one red and one blue)

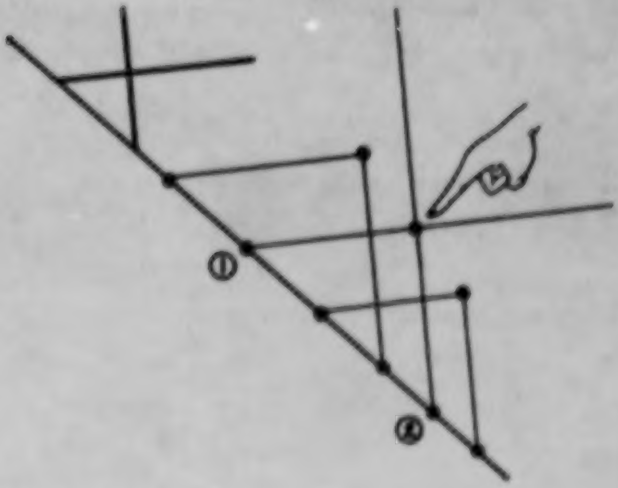
T: The double projection gives us a way to find two points on the line for every point above it. Suppose instead that we pick two points on the line. Can you find a single point above the line which will project to those two points?

Mark two points ① and ② as indicated here.

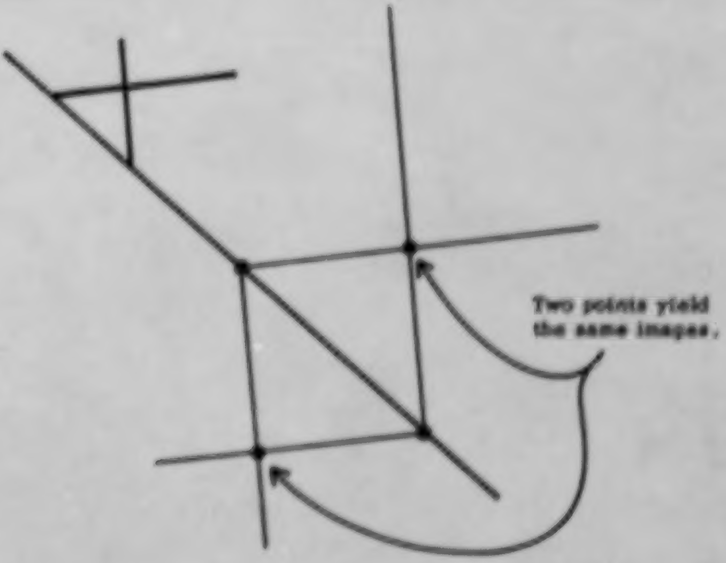


Encourage discussion and experimentation with the translator to locate the point above the line. Key observations are:

- to find the points which project onto ① parallel to the red segment, align the translator with the red segment, roll to ①, and draw a line;
- to find the points which project onto ② parallel to the blue segment, align the translator with the blue segment, roll to ②, and draw a line;
- the desired point is the intersection on these two lines, as shown in the next illustration.



NOTE : There is also a point below the line which corresponds to the same pair of points under the double projection. If anyone finds that point, verify its correctness and state that you are looking for a point above the original line.


Two points yield the same images.

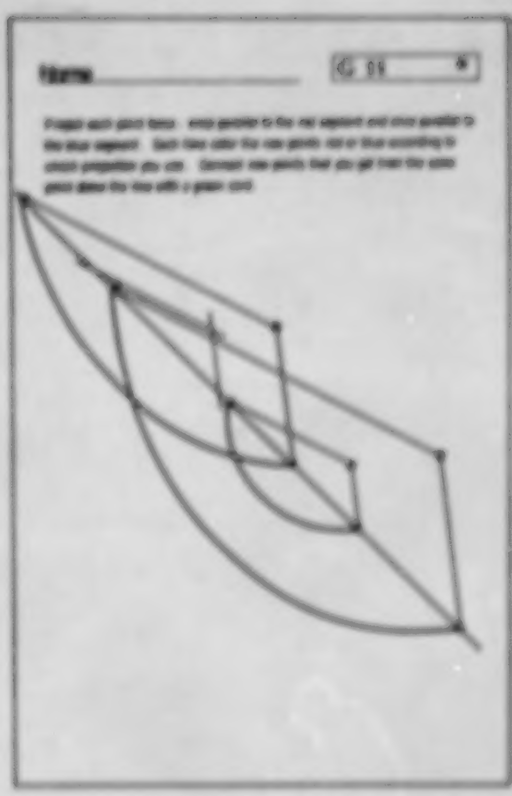
Choose a few more pairs of points on the line and ask students to find points above the line which project to them. Locate the points accurately with the translator.

T: For each point above the line, we can find two points on the line by the double projection. For every pair of points on the line, we can produce one point above the line. This gives us a way to choose two points on the line at random. If we first choose one point above the line at random, then we can perform the two projections. There are many ways to choose a point above the line at random.

Discuss any ideas the students have.

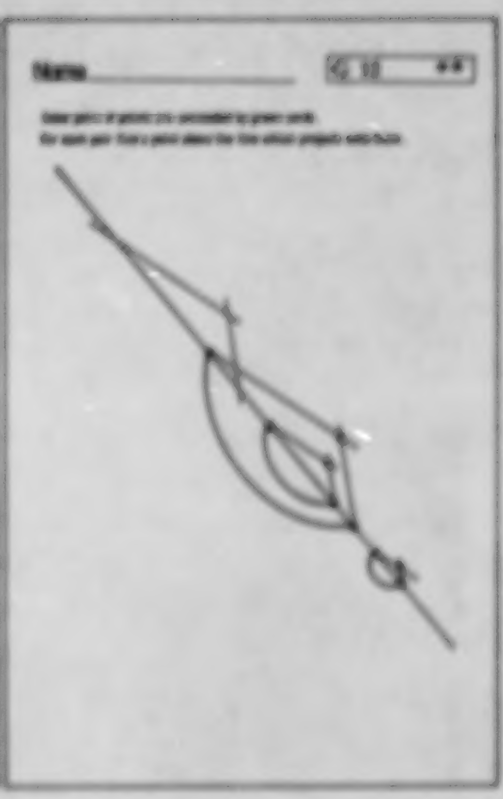
T: We could smear honey everywhere above the line and set a fly loose in the room. The fly will choose a point at random on which to land.

Provide individual translators and Worksheets G11 and G12 for students to practice the double projection.



Name

G 11 *



Name

G 12 **

G 11

1.10 Breaking a Stick #1

CAPSULE LESSON SUMMARY

Review the double projection method (from Lesson G1.9) for associating two points on a line with one point above the line. In the context of the problem of breaking a stick, construct a "honey" triangle with one side on the line and the third vertex found by associating the end points of the side with a point above the line. Find that only points inside the triangle project onto points of the side of the triangle on the original line. Practice the double projection with points inside the "honey" triangle and try to construct triangles with the three resulting pieces of the stick.

MATERIALS

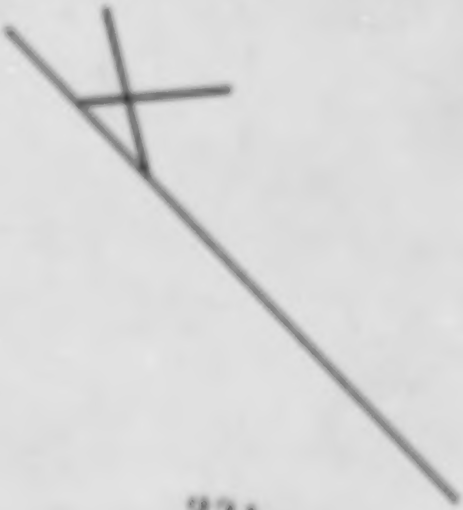
Teacher: Colored chalk; demonstration translator; chalkboard compass

Student: Colored pencils; individual translator; individual compass; Worksheet G11

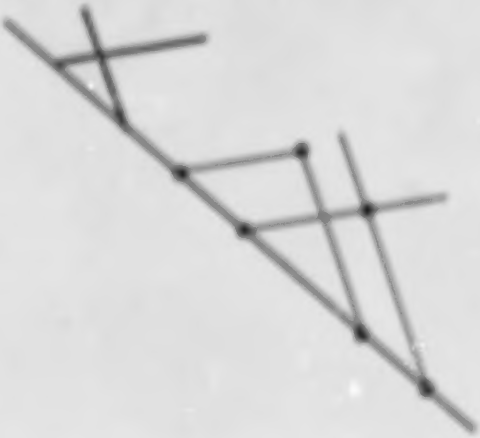
NOTE: See the note about translators with the materials list for Lesson G1.9.

DESCRIPTION OF LESSON

Restage the situation of Lesson G1.9 where a double projection method was presented. Draw a long line across the board and draw a red segment and a blue segment for projection directions as follows.

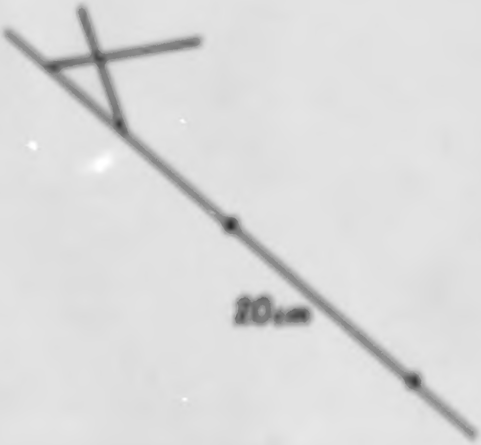


Review the double projection by choosing a point above the line and asking a student to use the translator to project it onto the line, first parallel to the red segment and then parallel to the blue segment. Repeat with one or two other points above the line. Next review the reverse process of choosing two points on the line and finding a point above the line which corresponds to this pair of points under the double projection.



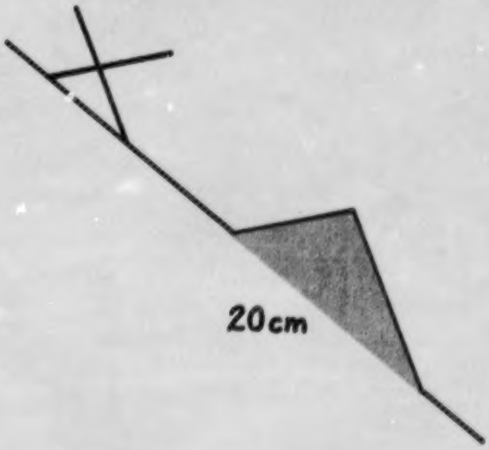
T: We can use a fly to choose a point above the line at random and then double project that point onto the line. But my stick is only 20 cm long so we do not need to consider the entire line.

Erase the practice points and projection lines and replace a portion of the line by a green segment to represent the stick.



T: Does every point above the line give us two breaking points for the stick?

Encourage discussion and experimentation. Sometimes choose a point above the line and check its images with the translator. Use yellow (representing honey) to mark points which do yield breaking points on the stick. Sometimes choose two points on the stick and find the point above the line which determines those points. Mark such points in yellow also. The object is to find the set of points whose images are on the stick. This set is shown in yellow in the following picture. One side of the yellow triangle is parallel to the red segment and the other is parallel to the blue segment. Use the translator to construct the triangle accurately. When the class agrees that the triangle includes exactly the points to be found, color your triangle yellow.



Experimentation with a few points on the sides of the triangles will lead to the observation that only points inside of the yellow triangle give two breaking points for the stick. Points on the sides of the triangle give one of the endpoints of the stick and only one breaking point. Also, the "top" vertex of the triangle gives the two endpoints of the stick and hence no breaking points at all.

T: We want the fly to choose breaking points for the stick. Where should we smear honey?

S: Inside the triangle.

T: This triangle is very important for our problem. Let's call it the "honey triangle." Now suppose we draw the red and the blue segments differently. How would that affect the honey triangle?

S: The honey triangle would take a different shape.

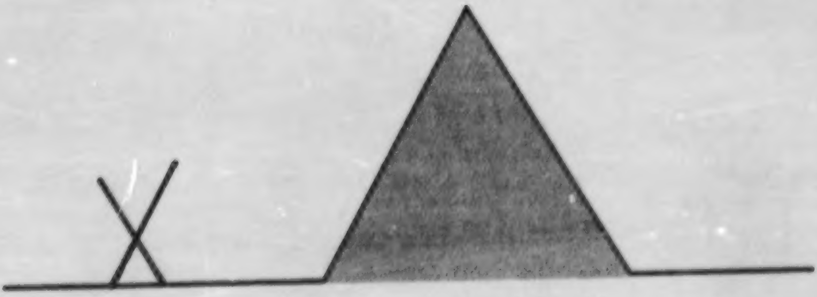
S: One side of the honey triangle is parallel to the red segment, so if you change the position of the red segment, you will have to change the position of that side of the honey triangle.

S: The same is true for the side parallel to the blue segment.

T: That's right. The sides of the honey triangle tell us the directions of the red and the blue projections. We'll change the triangle now and redraw the picture so that it is more convenient for our study. Let's make the sides of the honey triangle all the same length, and let's draw the stick (line) parallel to the floor.

Draw a green line segment to represent the stick. Use a compass to draw the equilateral triangle accurately. Set the radius of the compass to the length of the green segment. Placing the point of the compass in turn at each endpoint of the green segment, swing an arc above the green segment. Connect the endpoints of the green segment to the intersection point of the arcs to draw the desired equilateral triangle.

Use a translator to reestablish the red and the blue segments. Align the translator with each slant side of the triangle in turn and roll to the side to draw the red and the blue segments as shown here.



G 11

Now choose a point in the honey triangle and invite a student to perform the double projection onto the stick. Repeat with one or two more points until the students are secure in the reoriented picture.

T: Who remembers the problem of breaking the stick?

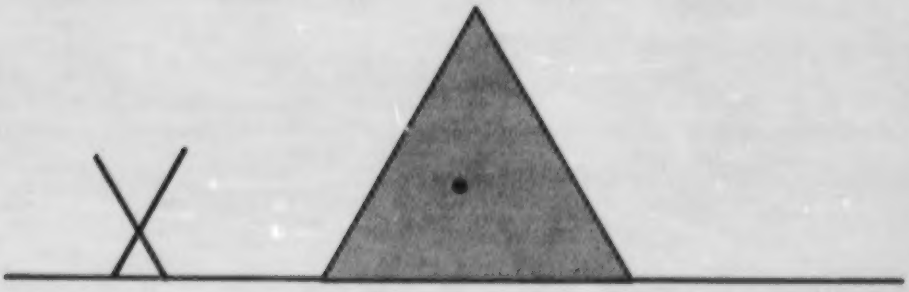
S: We want to break the stick at two points chosen at random and to try to form a triangle out of the three pieces.

S: We choose the two points at random by letting a fly land on the honey.

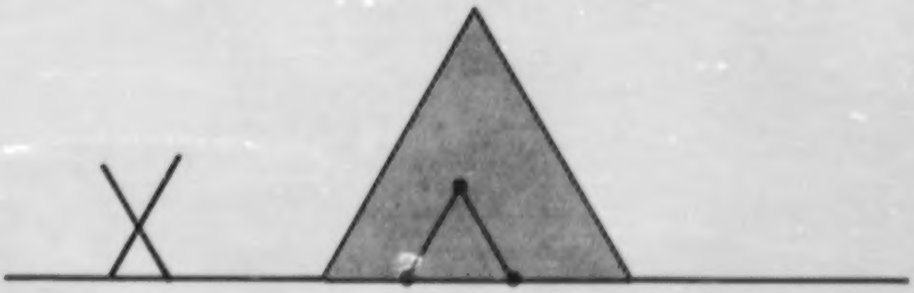
S: We want to find the probability of being able to make a triangle with the three pieces.

T: We need to find which points in the honey triangle give us a success. Let's try one.

Mark a point near the center of the triangle as shown here.

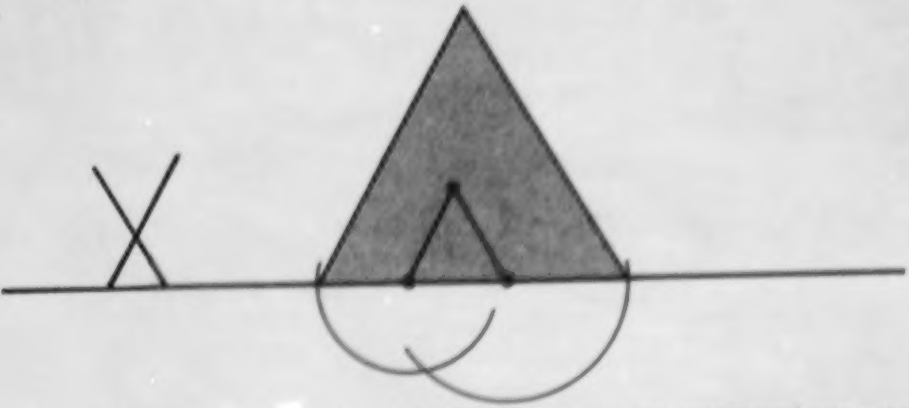


Call on a student to perform the red and the blue projections using a translator.

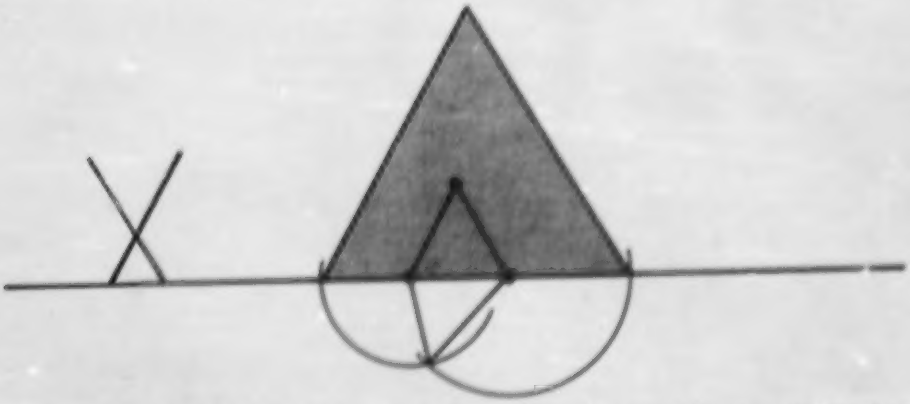


T: Now let's try to make a triangle with these three pieces. We need to use a compass.

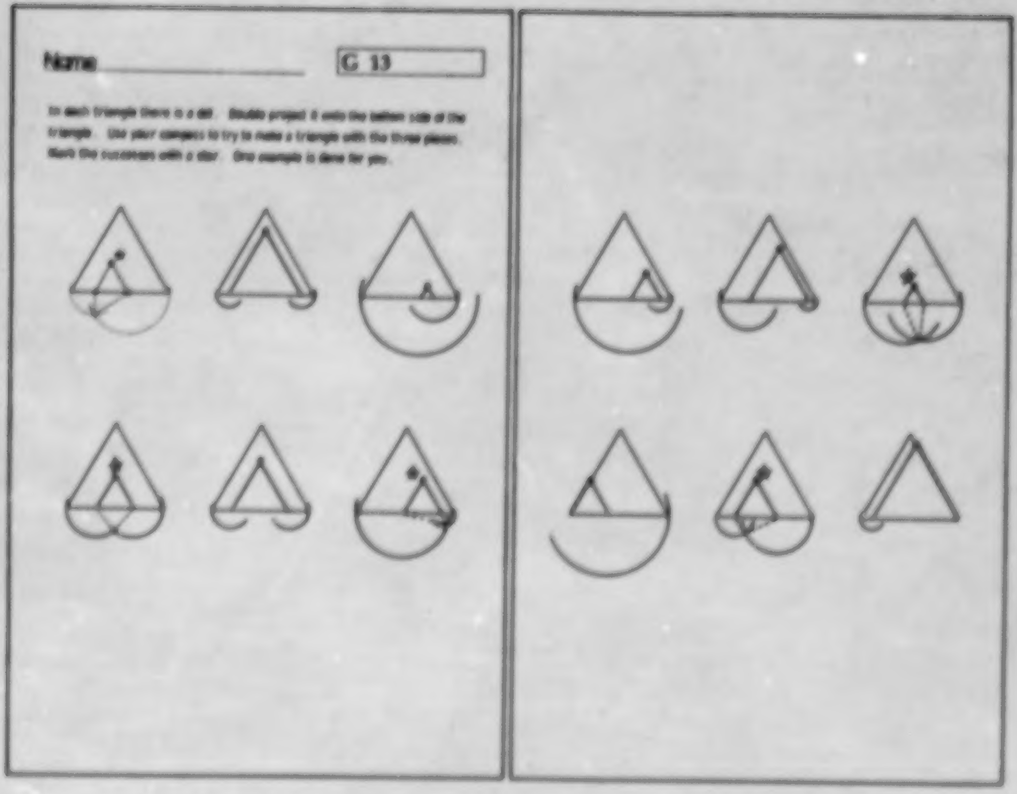
Of course, the construction might be carried out to the side of the honey triangle, but it can most conveniently be performed directly below the honey triangle with the middle piece left in place. Set the compass to the length of the left hand piece. Place the point of the compass at the red point and swing an arc below the honey triangle. Do the same for the right hand piece at the blue point.



In this case the two arcs intersect and the triangle can be completed successfully.



Provide Worksheet G13 for individual work with similar problems. Instruct students to use the slant sides of the triangles as the directions of the projections. Recall, if necessary, that in the picture on the board those sides are parallel respectively to the red and the blue segments. At the end of the lesson spend a few minutes comparing drawings and solutions.



Name

G 13

In each triangle there is a dot. Double project it onto the bottom side of the triangle. Use your compass to try to make a triangle with the three pieces. Mark the successes with a star. One example is done for you.

1.11 Breaking a Stick #4

CAPSULE LESSON SUMMARY

Recall the problem of breaking a stick as developed in Lessons G1.8, G1.9, and G1.10. Apply the double projection technique and the triangle inequality to solve the problem.

MATERIALS

Teacher : Colored chalk ; demonstration translator ; chalkboard compass

Student : Colored pencils ; individual translator ; individual compass

NOTE : See the note about translators with the materials list for Lesson G1.9.

DESCRIPTION OF LESSON

Draw a line segment about 40 cm long.

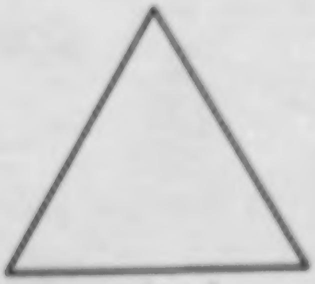
T : Here is a picture of my stick. Who can recall the problem of breaking a stick?

S : Break the stick at two points chosen at random. What is the probability that we can make a triangle with the three pieces?

T : How do we choose the two points at random?

S : By letting a fly land on the "honey" triangle.

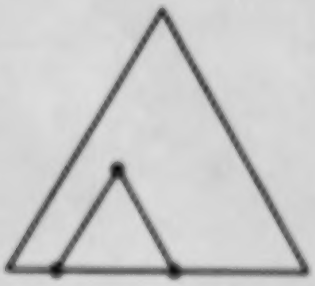
Use your compass to construct an equilateral triangle. Do not color it yellow.



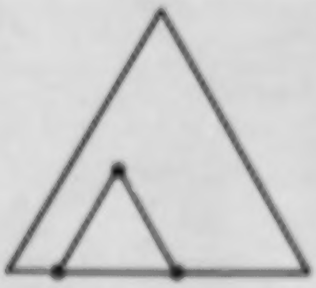
G 12

S: A fly landing in the triangle selects a point at random. Then we project the point onto the stick parallel to each side of the triangle to find the breaking points of the stick.

Choose a point and invite students to perform the projections using a translator at the board.



T: Let's color the three pieces.



T: What is a rule that tells us whether the three pieces will form a triangle?

S: Each piece must be less than half as long as the stick.

S: Each piece must be less than 10 cm long.

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Write this information in a box to the side of the drawing of the triangle.

● < 10

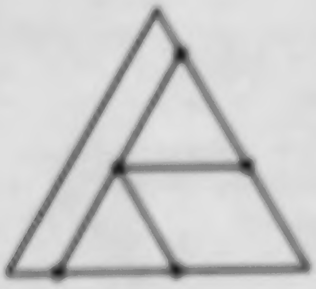
● < 10

● < 10

T: Each of the sides of the honey triangle is 20 cm long. Suppose that we put the stick along the right side of the triangle instead of the bottom. What would we do then?

S: We would project onto that side just as we did onto the bottom side.

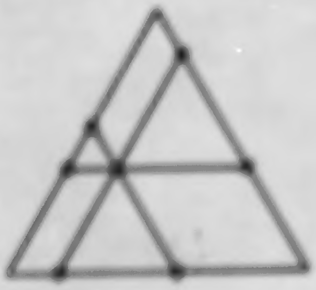
Invite students to add these projections to the drawing.



T: Could we put the stick along the left side of the triangle?

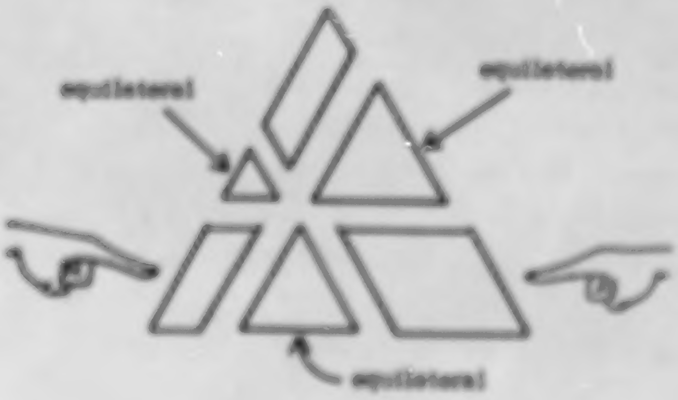
S: Yes, then we would project onto the left side.

Invite students to perform these projections.

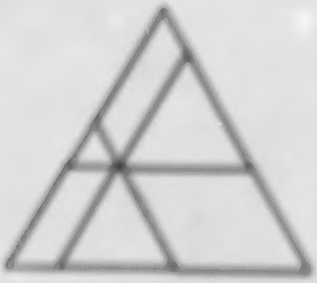


G 12

Encourage discussion of this picture. The object is color each segment red, blue, or green according to its length being equal to one of those already colored. Allow the class freedom to approach the coloring in any way or sequence. One good starting place is with one of the two parallelograms in the lower corners as shown in the "exploded" drawing here.



The key fact is the recognition of the three small equilateral triangles. If necessary, simply ask if anyone notices something interesting about one of these triangles. When a student suggests that all sides are of equal length, use the compass to verify the suggestion. Then color the edges appropriately. Repeat with the other small triangles as they are indicated by students. After coloring all of the edges by employing equilateral triangles and parallelograms, your drawing should be colored like the following.



Distribute Worksheet G14, translators, and compasses.

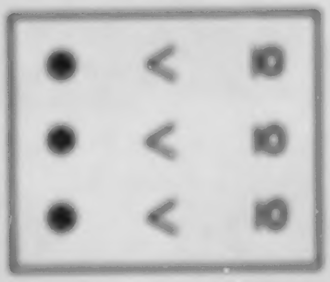
T: Each of you has a picture of the "honey" triangle.* Pick any point inside your triangle and project it onto each side just as we have done here. Then color the pieces red, blue, and green just as we have done in our picture on the board. All red pieces should be the same length, all blue pieces the same length, and all green pieces the same length.

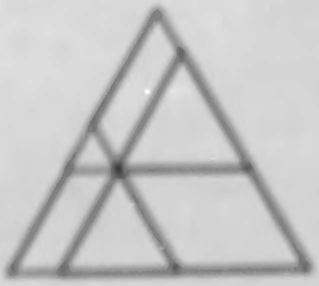
Allow some time for students to complete their drawings. Then compare drawings. Different choices of starting points will result in different lengths for red, for blue, and for green, but the segments will be in relatively the same positions.

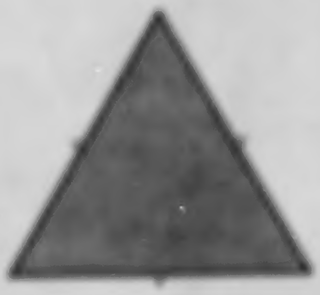
Draw another equilateral triangle on the board the same size as your original triangle and next to it. Color this triangle yellow to suggest honey and mark the midpoints of each side.

T: Here I have redrawn the honey triangle. I have marked the midpoints of the sides as is done on your worksheets. How long is the segment from a corner of the triangle to a midpoint?

S: 10 cm; just half of the full length of 20 cm.







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* Gray rather than yellow is used on the worksheet.

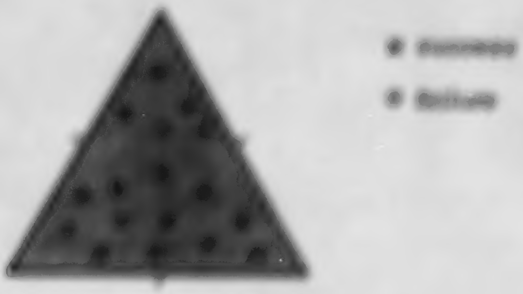
G 12

T: We need to find the points in the honey triangle that give us a successful breaking of the stick. What must be true about the red piece?

S: It must be less than 10 cm long.

T: Each of you set your compass to 10 cm. That's half of one side of the triangle on your worksheet. Now compare that length with the red piece in your picture. How many of you picked a point which made the red piece shorter than 10 cm?

Take a poll of the class and indicate many of the successes and failures on your yellow honey triangle. Begin with a mark for your own drawing.

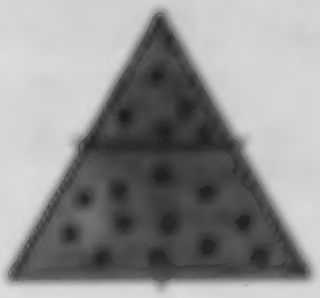


T: What can you say about the successes and the failures?

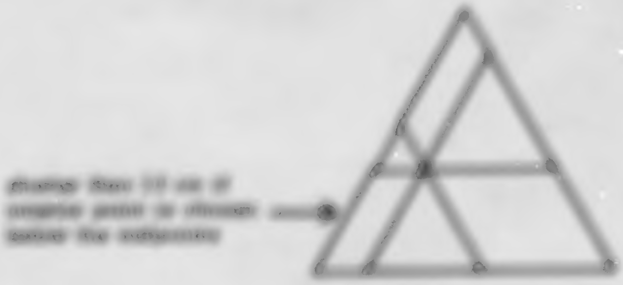
S: All of the successes are at the bottom and all of the failures are at the top.

S: If you connect the midpoints of the left and the right sides of the "honey" triangle, then all of the successes are below the connecting line.

Connect the indicated midpoints.

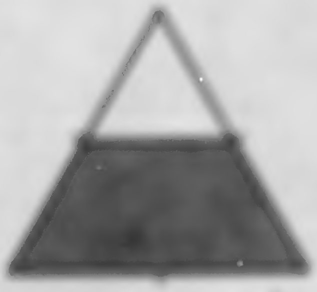


Refer to your original triangle to observe that for the red pieces to be shorter than 15 cm., the chosen point must be below the line connecting the midpoints. This is most easily recognized by considering one of the red segments along a side.


shorter than 15 cm if
original point is chosen
below the midpoints

T: Let's remove the beans from the area which we now know gives failures.

Do so, and also remove the marks for successes and failures. Then repeat the coloring of the lower section.

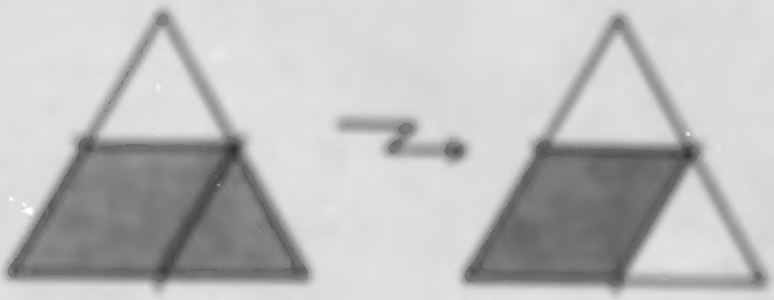


G 12

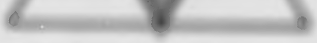
T: That takes care of the red piece. What about the blue piece?

S: It too must be shorter than 10 cm.

Again poll the class and record successes and failures in producing a blue segment shorter than 10 cm. Successes will be to the left and failures to the right of a dividing line formed by connecting two midpoints as shown below. Again remove the honey from the area yielding failures.



Repeat the discussion once more for the green piece. The final result is shown in the next picture.



T: Now at last we have found all of the points which give us three pieces to form a triangle. For a success, where must the fly land?

S: In the center triangle.

S: All four small triangles have 10 cm long sides.

S: All four small triangles are the same size.

T: What is the probability of our getting those pieces that make a triangle?

S: The fly gives us a success by landing anywhere in the center triangle, is one out of four triangles.

S: All of the triangles are the same size, so the probability is one chance out of four or $\frac{1}{4}$.

2

A GRID GAME

This chapter presents a game using the coordinate grid and thereby gives practice in corresponding points of the grid with ordered pairs of numbers. The game involves marking the four corners of a square with a random procedure for deciding which corners may be marked. It provides informal analytic geometry experience and develops spatial perception in still another kind of activity. Although the game is introduced in just one lesson, it could be played collectively or with a small group of students at other times during the year.

NOTE ON GRIDS

In this chapter, as well as in many lessons in the other parts of the CSMP Intermediate Grades curriculum, you will need a large grid board for demonstrations. A portable chalkboard with a permanent grid is convenient and efficient. Squares with eight centimeter sides are ideal. However, if such a grid board is not available to you, there are several alternatives.

Printed grids on large sheets of paper* are provided with full classroom sets of materials for CSMP Mathematics for the Intermediate Grades, Part V. They can be taped to the board and discarded after use. Or one could be mounted on a board and covered with clear contact paper to provide a protective surface. Erasable wax crayon can then be used for drawings.

Alternatively, use the grid transparency provided in the classroom set of materials and an overhead projector to "display" a grid on the chalkboard.

Another possibility is to carefully draw a grid on the chalkboard each time. The disadvantage of this technique is that inevitably the grid lines are partially erased each time a drawing is altered. However, one can make a tool for drawing grids. Use a music staff drawer with chalk in the first, third, and fifth positions. Or proceed as follows: clip four or five pen caps to a ruler at six to eight centimeter intervals and secure them with tape. Pen caps are usually of a diameter to hold a piece of chalk securely.

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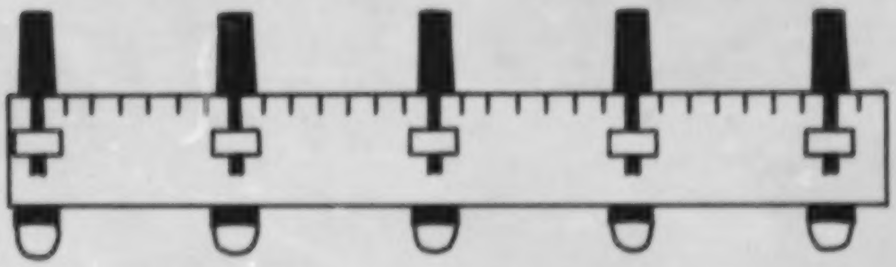
* These printed grids are packaged under the title "Robot Walk Grid". The title is from lessons in the CSMP Kindergarten Program and has nothing to do with the lessons in this chapter.



CAPSULE LESSON SUMMARY

Section 2.1: Correspond the corner points of a grid with ordered pairs of integers. Mark four points on the grid so that they are the corners of a square. Using the result of rolling specially labeled cubes to designate ordered pairs, locate points in the grid until the four corners of a square have been marked. Play a game using four cubes and the idea of marking points on the grid which can be named from the results of rolling the cubes. The first person to mark the four corners of a square is the winner. (Worksheet G15)

2.1 A Grid Game

CAPSULE LESSON SUMMARY

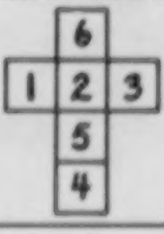
Correspond the corner points of a grid with ordered pairs of integers. Mark four points on the grid so that they are the corners of a square. Using the results of rolling specially labeled cubes to designate ordered pairs, locate points on the grid until the four corners of a square have been marked. Play a game using four cubes and the idea of marking points on the grid which can be named from the results of rolling the cubes. The first person to mark the four corners of a square is the winner.

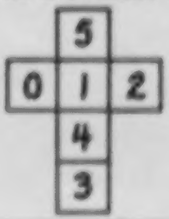
MATERIALS

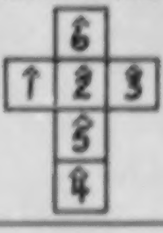
Teacher: Grid board; four specially marked cubes; magnetic checkers (optional)

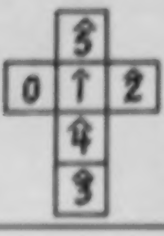
Student: Pencil; Worksheet G15

NOTE: See the NOTE ON GRIDS in the introduction to this chapter. Prepare four cubes (dice or other cubes which roll well) by pasting labels on their faces as indicated in these cube cutouts.



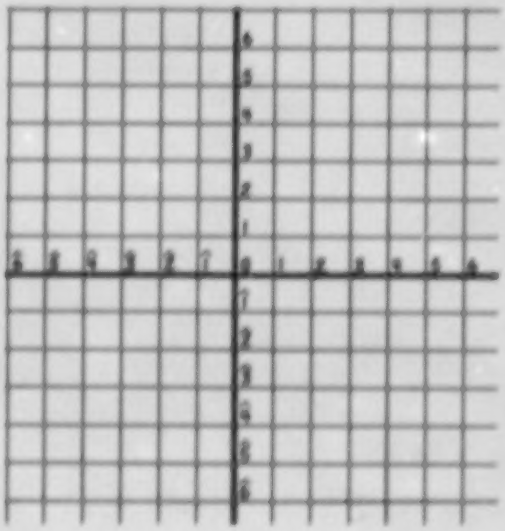



$\hat{6}$
$\hat{1}$
$\hat{2}$
$\hat{3}$
$\hat{5}$
$\hat{4}$


$\hat{5}$
0
$\hat{1}$
$\hat{2}$
$\hat{4}$
$\hat{3}$

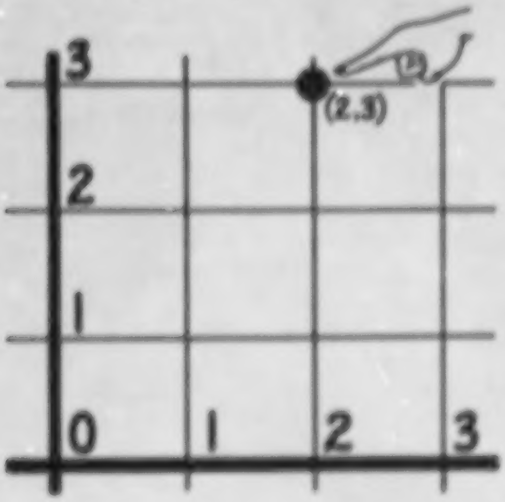
DESCRIPTION OF LESSON

Draw a horizontal line and a vertical line on your grid board and label points as follows on those lines.


6
5
4
3
2
1
$\hat{6}$
$\hat{5}$
$\hat{4}$
$\hat{3}$
$\hat{2}$
$\hat{1}$
0
1
2
3
4
5
6
$\hat{1}$
$\hat{2}$
$\hat{3}$
$\hat{4}$
$\hat{5}$
$\hat{6}$

G 7

T: Here is a grid with some points labeled. Each point on the grid corresponds to a couple (or ordered pair) of numbers. One such couple is (2,3). (Write it on the board.) The first number in the pair tells us how far to move right or left from 0. In this case, move two to the right. (Trace with your finger on the grid.) The second number in the pair tells us how far to move up or down from 0. In this case we move three up. (Trace with your finger, mark the point, and label it.



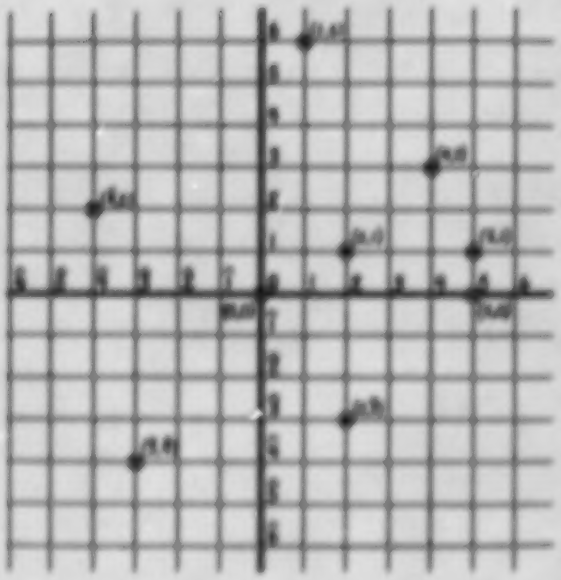
Suggest several other pair of numbers and have students locate the points on the grid. Reinforce the rules:

- Start at 0 (or (0,0), the intersection of the axes).
- First number is for right or left.
- Second number is for up or down.
- Positive numbers signify right or up.
- Negative numbers signify left or down.

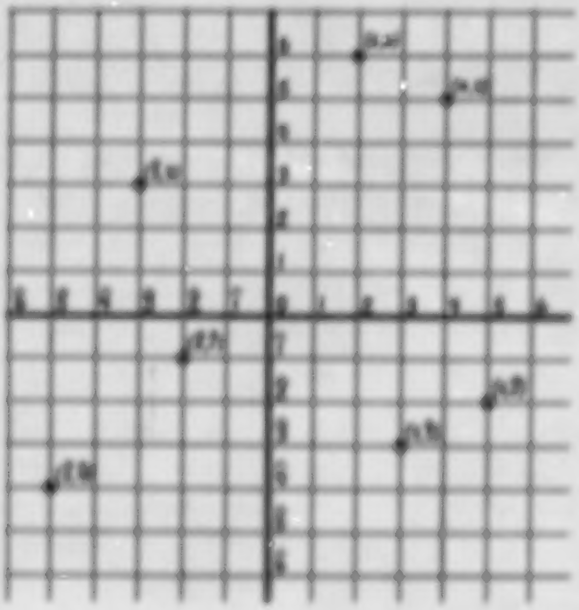
Continue until several points are located and the class is secure in the technique. For example, you may locate points for the pairs indicated in the next illustration.

$(2,1)$ $(4,3)$ $(1,6)$ $(2,\hat{3})$

$(5,1)$ $(5,0)$ $(\hat{4},2)$ $(\hat{3},\hat{4})$ $(0,0)$



Next, mark a few points and ask students to give the corresponding ordered pairs of numbers. For example, you could use the points indicated in the following illustration.



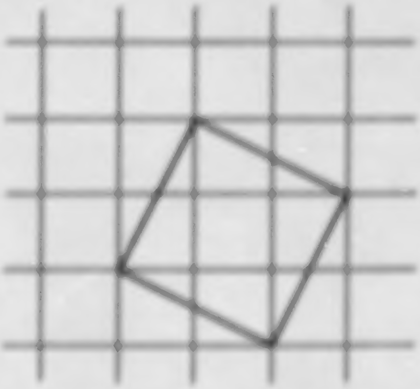
G 7

Exercise 2

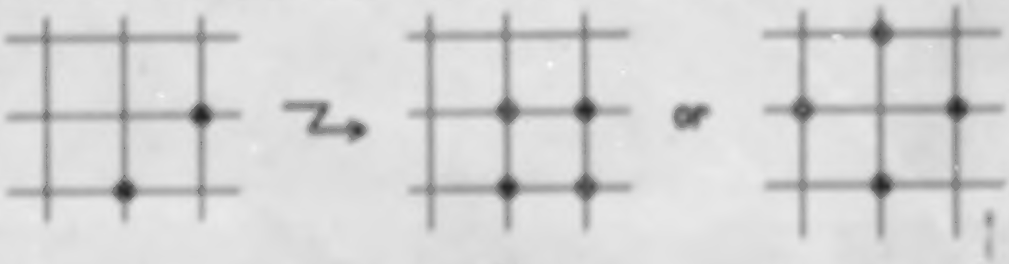
If your grid board is magnetic, use Minicomputer checkers for convenience in marking points.

T: Here are four checkers. Who can put them at points on the grid to mark the four corners of a square?

Ask several students to do this. Expect some errors, especially in suggesting the corners of a rectangle that is not a square. Encourage class discussion to correct any errors. There are many, many squares which could be shown on the grid using only points at the intersections of grid lines. (In fact, 2,366 squares can be formed on the grid used in the illustrations here.) In most classes, the last squares to be discovered are those skew to the grid lines. For example,



To exercise the students perception of squares on the grid and to introduce skewed squares, if necessary, devote a few minutes to a game in which you mark three points for three corners of a square and ask a student to find the fourth corner. Then change the game by only locating two points and requiring students to find two others to complete a square. Note that sometimes there will be two solutions to the latter problem. For example,

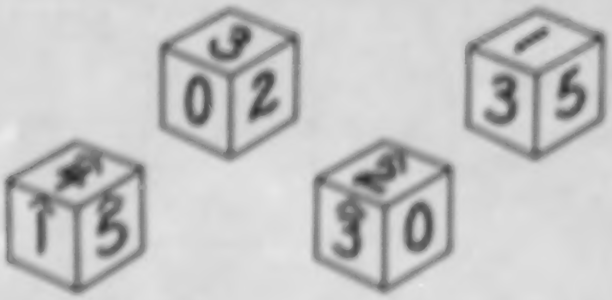


Exercise 3

Display the cubes.

T: I have here four cubes. One is labeled with "1", "2", "3", "4", "5", and "6". Another is labeled with "0", "1", "2", "3", "4", and "5". A third is labeled with "$\hat{1}$", "$\hat{2}$", "$\hat{3}$", "$\hat{4}$", "$\hat{5}$", and "$\hat{6}$". And the last is labeled with "0", "$\hat{1}$", "$\hat{2}$", "$\hat{3}$", "$\hat{4}$", and "$\hat{5}$". We shall roll all four cubes and read the four numbers that come up.

Roll the cubes on a desk at the front of the room. Let a student read the numbers as you write them on the board.



S: $\hat{4}$, 3, $\hat{2}$, and 1.

T: We can use these cubes to pick points on the grid. Each time we roll the cubes, we choose two of them to name a point and then mark it on the grid. Which points could we mark with this roll?

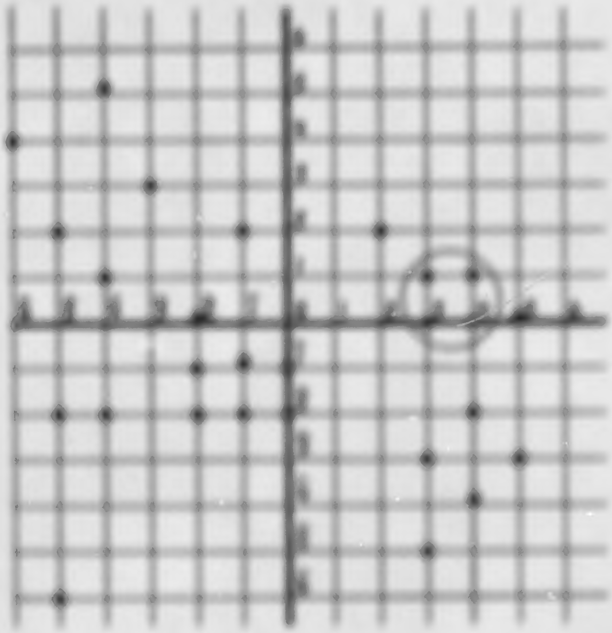
There are twelve points which could be marked with the roll given in the example above.

$(\hat{4}, 3)$	$(3, \hat{4})$	$(\hat{2}, \hat{4})$	$(1, \hat{4})$
$(\hat{4}, \hat{2})$	$(3, \hat{2})$	$(\hat{2}, 3)$	$(1, 3)$
$(\hat{4}, 1)$	$(3, 1)$	$(\hat{2}, 1)$	$(1, \hat{2})$

G 7

T: After picking a point, we roll again and mark another point. The object is to see how quickly we can mark the four corners of a square.

Continue rolling, picking, and marking points until the corners of a square are marked. Each time the class should check carefully that the point chosen is correct in that the two numbers of its ordered pair show on the cubes. The result of one collective game from our development classes is shown below. The game ends when the class observes that the four corners of a square are marked. (See the circled section of the picture.) In fact, in this game the corners of another square ($(^-5,^-2)$; $(^-5,2)$; $(^-1,2)$; $(^-1,^-2)$) had been marked earlier but went unnoticed. Also, there were several possibilities for completing squares. Having legally marked any of the five red points might also have finished the game.



If the class is confident in the procedure of the game, proceed to the next exercise. Otherwise, play another collective game.

__Exercise 4__

Distribute Worksheet G11.

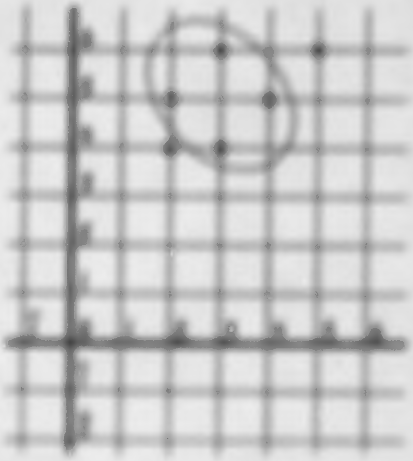
T: Now let's play the game individually. I will roll the cubes and write the results of each roll on the board. Each time you may choose two of the numbers to locate a point on your grid. You may not change a point once you have marked it. Let's see who will mark the corners of a square first.

Play the game. List the results of each roll in a column on the board. Continue until a student announces having completed a square. Ask the student to mark the corners of the square on the grid board for checking by the class. Verify the legality of each of the suggested corners by comparing the ordered pairs for the corners against the lists of rolls.

Repeat the game as long as interest sustains.

Here is the tabulation of rolls and the winning plays of a game in one of our development classes.

roll	1	2	3	4	5	6
	4	5	2	3	2	3
	5	6	5	4	4	6
	1	2	1	0	1	2
	3	6	3	5	3	5
points chosen	(4,5)	(5,6)	(2,5)	(3,4)	(2,4)	(3,6)



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Lesson Seven Pt. 2 – [illegible] [illegible]
Lesson Eight Pt. 2 – [illegible] [illegible]
Lesson Nine Pt. 4 – [illegible] [illegible]

[illegible]

PROBABILITY AND STATISTICS

INTRODUCTION

There are several reasons why people study mathematics. It is an interesting and challenging intellectual exercise often studied for its own sake. But it is also a supremely useful topic of study in that it has applications outside it and in other disciplines, as well as more generally in understanding the world around us. Some areas of mathematics are more easily applied than others, and the areas which particularly concern us in this strand are towards the easy end of the spectrum, both because of the small amount of background mathematical knowledge required before meaningful applications can be made, and because it is usually fairly easy to see the link between the mathematical model and the "real" situation being modeled. For these reasons, it would be difficult to overstate the importance in today's world of probability and statistics and their applications to problems and situations within and outside the discipline of mathematics.

The lessons of this strand deal with systematic methods of analyzing games of chance and of finding numerical probabilities for outcomes or events. Discussions which involve us in finding probabilities of certain events give rise to combinatorial problems and give support to our work with functions in the World of Numbers strand. Furthermore, the experiments described in the lessons of Chapter 1 exploit relations on the whole numbers and number stories involving these relations, so there is a further connection with the N-lessons and with the L-lessons. The lessons of Chapters 2 and 3 make use of a particular [illegible] method for solving problems to find the most probable outcome. In some situations the information provides the basis for making a decision.

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RANDOM NUMBER STORIES

The lessons of this chapter are concerned primarily with combinatorics problems. Initially an experiment is presented in which random selections are made from three (or five) boxes containing mathematical symbols (for numbers, for operations, and for relations) and number stories result. Several trials of the experiment are performed to observe some possibilities of outcomes and to initiate discussions of predicting the truth or falsity of a resulting number story. The overriding problem of determining the probability of getting a true statement is posed. Then experimentation is abandoned and attention focuses on finding the number of possible statements which can arise from the situation (the main combinatorial principle being the "multiplication rule") and on using organizational methods (tables) to sort out which statements are true.

As well as posing an interesting probability question, the "boxes" experiments provide a rich setting in which to recall previous concepts.

- In the first of the three lessons, the students look again at some relations which play a vital role in the Table Game (Languages of Strings and Arrows strand).
- In the second and third lessons, the students review the arithmetic of fractions which supports work in the World of Numbers strand.

CAPSULE LESSON SUMMARIES

Section 1.1: Make random selections from each of three boxes containing symbols for numbers and for relations and examine the resulting number stories. Determine how many possible number stories there are and how many of them are true. Find that the probability of getting a true number story is 35/100 or 35 out of 100. (Optional) In a similar situation, decide which numbers to put into the boxes so as to have the largest (smallest) possible probability of getting a true number story. (Worksheets P01, P02)

Section 1.2: Make random selections from each of five boxes containing symbols for numbers, for operations, and for relations, and examine the resulting number stories. Find true and false number stories for each of the operations. Discover ways to complete number stories with a number from the fifth string (box) so that the results are true. Discuss how to find the probability of getting a true number story.

Section 1.3: Review the "five boxes" experiment introduced in Section 1.2. Extend the exploration and discussion to determine how many possible number stories there are and how many of them are true. Find that the probability of getting a true number story is 33/100, or about 1 out of 3.
(Worksheet P03)

1.1 Random Number Stories #1

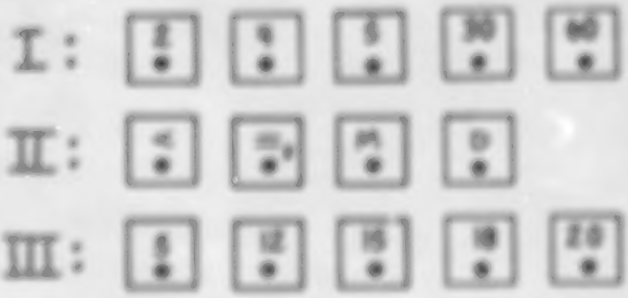
CAPSULE LESSON SUMMARY

Make random selections from each of three boxes containing symbols for numbers and for relations and examine the resulting number stories. Determine how many possible number stories there are and how many of them are true. Find that the probability of getting a true number story is 38/100 or 38 out of 100. (Optional) In a similar situation, decide which numbers to put into the boxes so as to have the largest (smallest) possible probability of getting a true number story.

MATERIALS

Teacher: Three boxes; pieces of paper (about 10cm by 10cm)
Student: Paper; colored pencils; Worksheets P01 and P02

NOTE: Prior to presentation of this lesson, prepare three boxes with labels "I", "II", and "III". Into the boxes, put pieces of paper marked as indicated below and folded so as to hide what is written on them.



DESCRIPTION OF LESSON

Exercise 1

Write these number stories on the board.

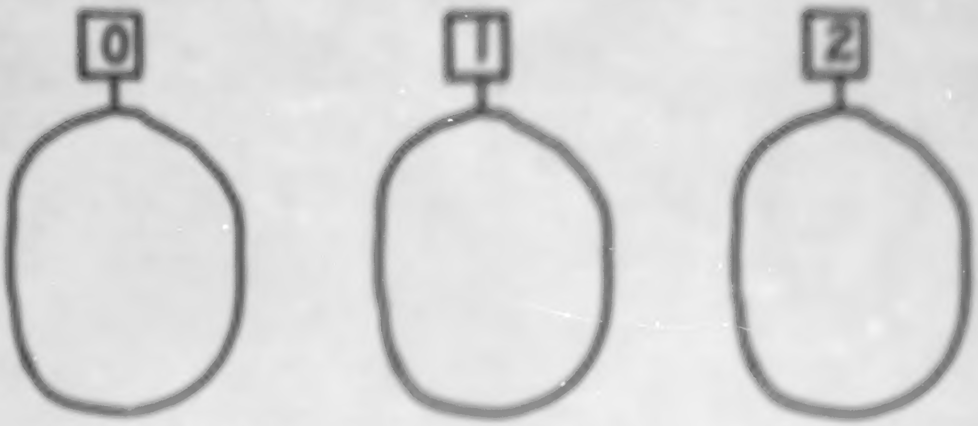
$$4 =_3 15$$

$$5 =_3 20$$

T: Do you remember the relation $=_3$ (read "equals modulo three")?

P 1

Let the students recall as much as they can about this relation. Draw three strings on the board.



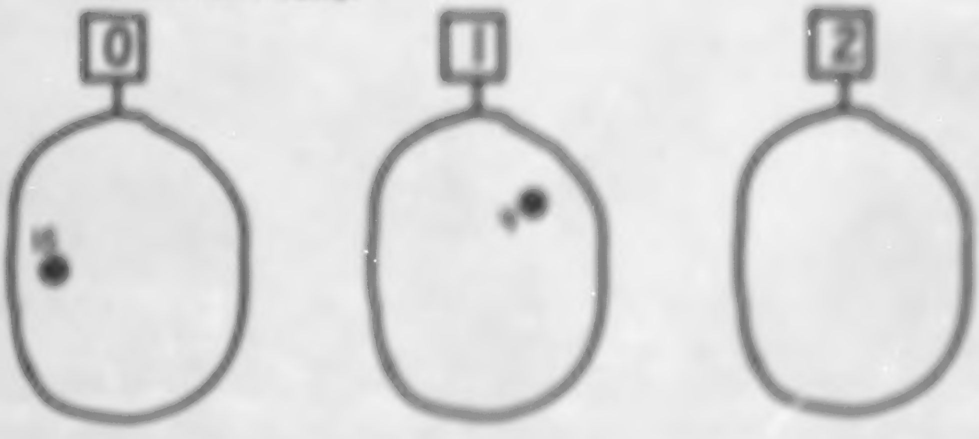
T: When we discuss $=_3$, what are these three strings for?

S: The 0-string is for multiples of 3, the 1-string is for numbers which are 1 more than a multiple of 3, and the 2-string is for numbers which are 2 more than a multiple of 3. Two numbers are equal modulo 3 if and only if they are in the same string.

If an explanation of $=_3$ is not forthcoming, explain the relation yourself.

T: Are the number stories I write on the board true or false?

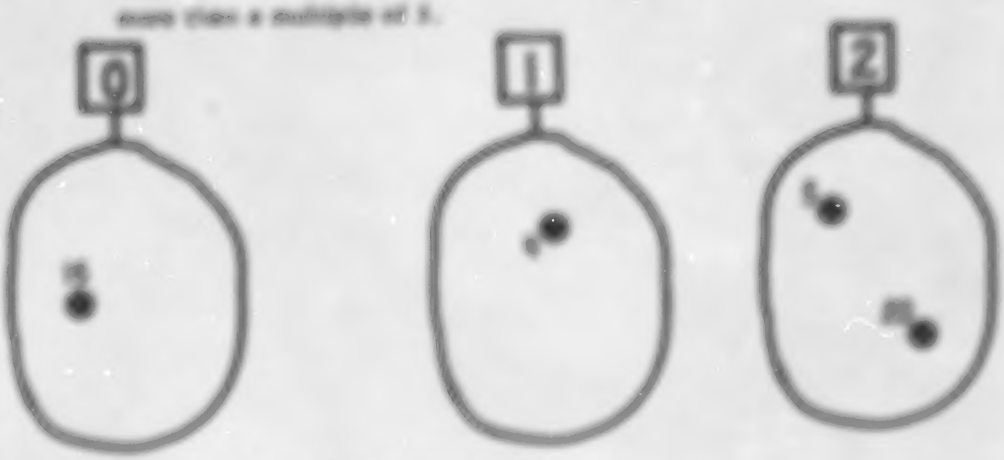
S: "4 $=_3$ 15" (read: four is equal modulo three to fifteen) is false because 4 and 15 are not in the same string; 4 is in the 1-string and 15 is in the 0-string.



P 1

Write "F" near the number story "4 $=_3$ 15".

T: "5 $=_3$ 20" is true because 5 and 20 are both in the 2-string; each is 2 more than a multiple of 3.



Write "T" near the number story "5 $=_3$ 20".

Write these problems on the board.

4 $=_3$ ☐ T   30 $=_3$ ☐ F

☐ $=_3$ 18 T   ☐ $=_3$ 20 F

Invite students to fill in the boxes to make each number story true or false as indicated. When necessary, put numbers in the strings to check solutions. Many solutions are possible; however, just one solution for each problem is provided here.

4 $=_3$ $\boxed{13}$ T   30 $=_3$ $\boxed{7}$ F

$\boxed{12}$ $=_3$ 18 T $\boxed{10}$ $=_3$ 20 F

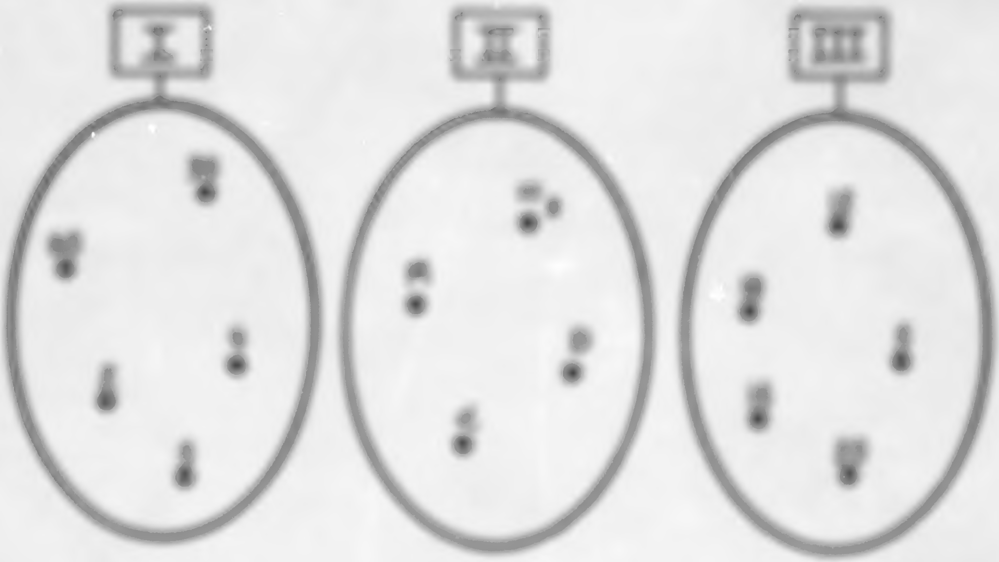
P 1

Exercise 1

Due to the randomness of the activities in this exercise, be prepared to adapt the discussion to the results in your classroom.

T: In these three boxes there are pieces of paper, each with a symbol written on it and folded so that the symbol is hidden. I will draw a picture to show what is in each box.

As you draw this string picture on the board, read the symbols and describe what is in each box (string).



Be certain that the students understand that "D" abbreviates "is a divisor of", "M" abbreviates "is a multiple of", and "<" abbreviates "is less than". If necessary, give an example of each of these relations.

T: Now we will perform trials of an experiment with these three boxes to make number stories. I'll shake each of the boxes and one of you will select, without looking, one piece of paper from each box.

Choose a student to make a selection, and ask the student to read aloud what is on each piece of paper immediately after selecting it. Progressively record the resulting number story on the board. For example,

5 M 20

T: Is this number story true or false?

S: False; 5 is not a multiple of 20.

5 M 20 F

Put the pieces of paper back into their original boxes. Call on students to perform two or three more trials of the experiment. Record the resulting number stories; for example,

5 M 20 F

4 D 12 T

2 =.20 T

Choose a student to perform another trial of the experiment. Proceed as before, except stop the student after papers are selected from Box I and Box II. A student might select "4" and "D".

4 D

T: Wait a moment. Before you select from the third box, can we predict whether the resulting number story will be true or false?

S: Both are possible; "4 D 20" is true but "4 D 15" is false.

P1

T: How could we get a true number story?

S: If 12 or 20 is selected from the third string, because 4 is a divisor of 12 and 4 is a divisor of 20.

T: How could we get a false number story?

S: If 5, 15, or 18 is selected.

T: Which is more likely, a true or a false number story?

S: False, because there are three chances for a false number story and only two chances for a true number story.

T: What is the probability of a true number story?

S: $\frac{2}{5}$ or 2 out of 5.

Permit the student to select from the third box and record the result on the board. For example, 20 might be selected.

4 D 20 T

Choose a student to perform another trial of the experiment except ask the student to make the selections starting with the third bag and to stop after selecting from bag III and bag II. The incomplete result might be:

< 12

T: Which numbers in the first string (bag) would give a true number story?

S: [illegible], [illegible], or [illegible].

P1a

T: What is the probability of getting a true number story?

S: [illegible] or [illegible] out of [illegible].

[illegible] the student to [illegible] from the four [illegible] and record the result on the board. For example:

$$60 < 12 \quad F$$

Exercise 2

Draw this table on the board. In each position put a relation symbol to be part of a number story.

True	False
=	=
>	>
<	<
[illegible]	[illegible]

T: Who can give an example of a true number story we could get using "="?

S: "[illegible]".

Draw a true number story using "=" in the table.

F1

[illegible]

60 M 12 T

12 M 60 F

[illegible]

[illegible]

True	False
60 M 12	30 M 20
5 D 15	30 D 5
$2 < 5$	$30 < 18$
$5 =_5 20$	$6 =_5 5$

[illegible] [illegible]

[illegible] [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

P1

D	6	12	15	18	20
2					
4					
5					
30			●		
60					

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

D	6	12	15	18	20
2					
4					●
5		F			
30			F		
60					

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

F 1

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

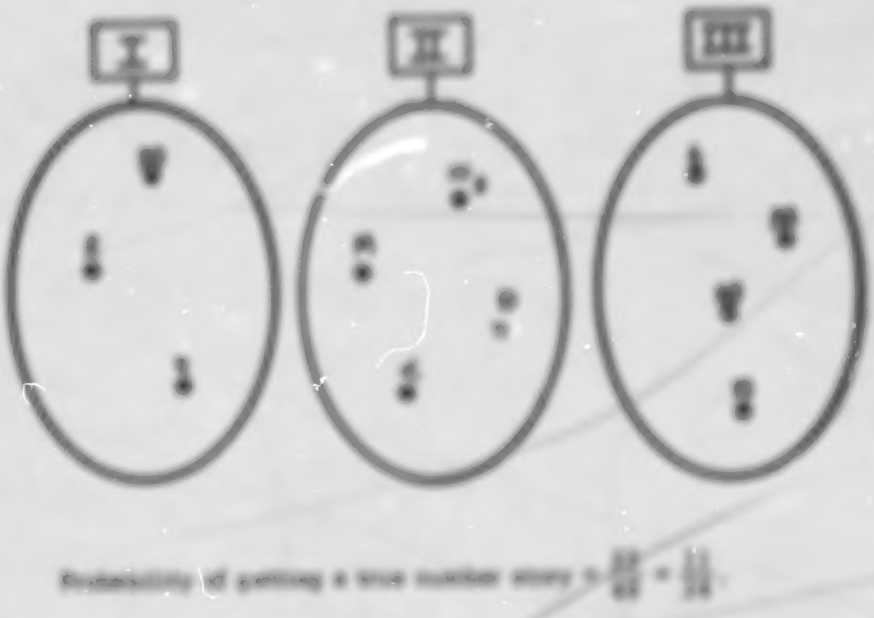
[illegible]

T: Imagine the three boxes. In the second box, we put the same relation symbols as before. In the first box we put three numbers, and in the third box we put four numbers different from those in the first box. Your problem is to decide which numbers to put in the first and third boxes so as to have the largest possible probability of getting a true number story.

Instruct students to copy the picture on their papers, then to label the dots with positive whole numbers (all different in each string) so as to have as many true number stories as possible.

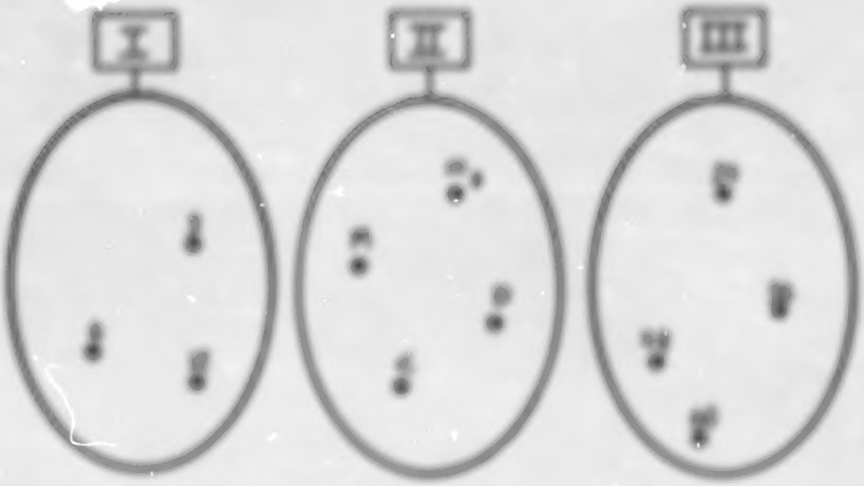
Let students work independently. When a labeling is suggested, direct students to calculate the probability of getting a true number story. Encourage using tables to help count the true number stories. On the board record some of the possibilities students find.

There are two possible solutions.



Probability of getting a true number story = [illegible] = [illegible].

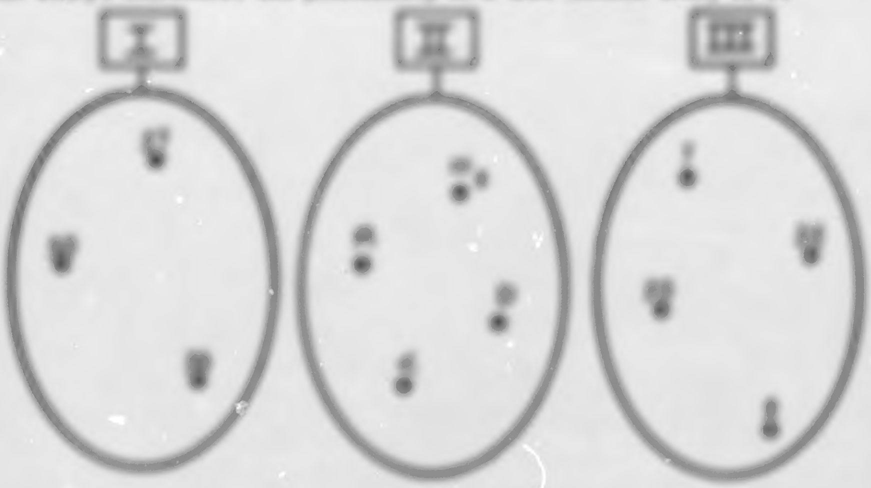
P 1

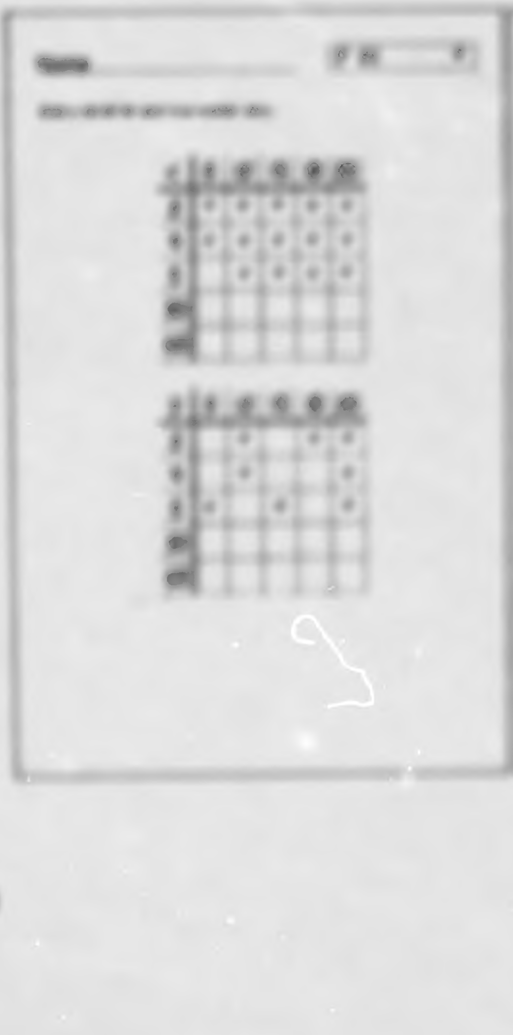


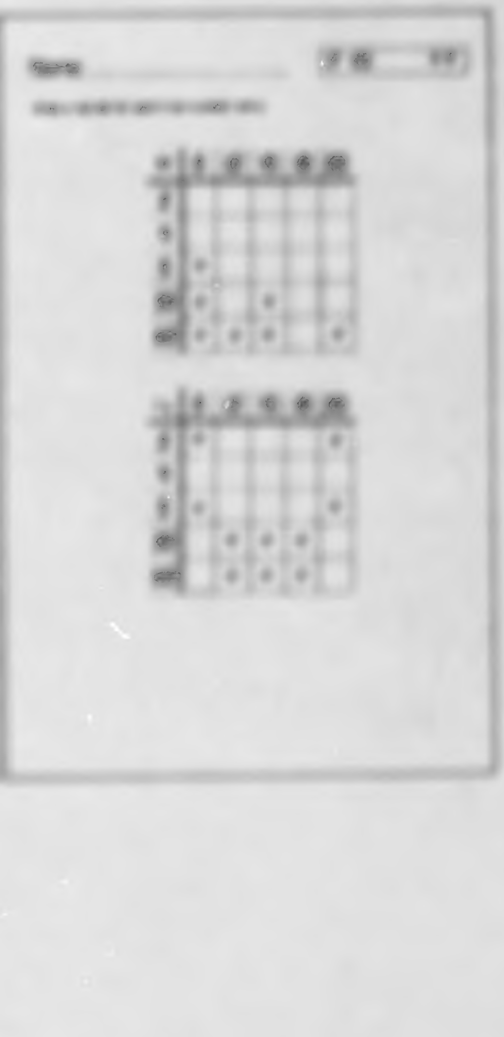
Probability of getting a true number story = $\frac{48}{64} = \frac{3}{4}$.

NOTE: One strategy for obtaining the largest possible probability of getting a true number story, that is, $\frac{3}{4}$ (3 out of 4), is to select numbers so that all the number stories using "<", "≤", and "≠" are true. This causes all the number stories using ">" to be false; therefore, three out of four number stories will be true. This strategy was used in the solution above. Many such solutions are possible.

Students who finish this problem quickly could be asked to label the dots so as to have the smallest possible probability of getting a true number story. For this problem, a labelling such as the following makes it impossible to get a true number story; therefore, the probability of a true number story is 0.







P 7

[illegible] Random Number Stories [illegible]

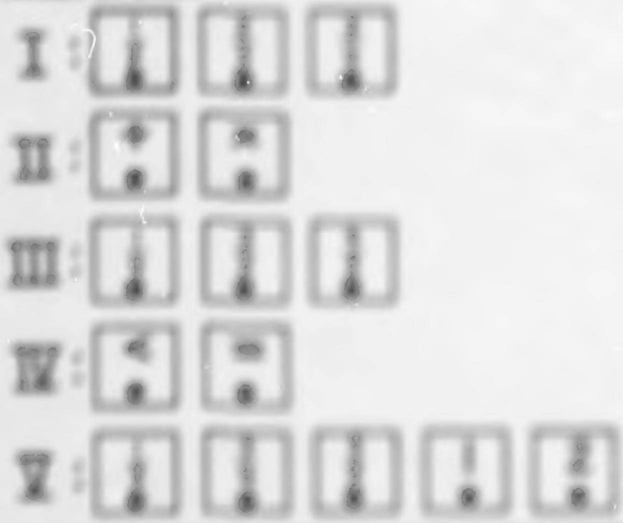
CAPSULE LESSON SUMMARY

Make random selections from each of five boxes containing symbols for numbers, for operations, and for relations, and examine the resulting number stories. Find true and false number stories for each of the operations. Discover ways to complete number stories with a number from the fifth string (box) so that the results are true. Discuss how to find the probability of getting a true number story.

MATERIALS

Teacher: Five boxes; pieces of paper (about 15 cm by 10 cm)
Student: Paper; pencil

NOTE: Prior to presentation of this lesson, prepare five boxes with labels "I", "II", "III", "IV", and "V". Into the boxes, put pieces of paper marked as shown below and folded so as to hide what is written on them.



DESCRIPTION OF LESSON

Due to the randomness of the activities in this lesson, be prepared to alter the discussion to the results in your classroom. For adding, multiplying, and comparing fractions, use the explanations from previous lessons which you feel are most appropriate for your students. The lesson description includes a variety of possible approaches which the students might suggest.

Exercise [illegible]

T: In these five boxes there are pieces of paper, each with a symbol written on it and folded so that the symbol is hidden. I will draw a picture to show what is in the boxes.

T: Now I'll shake each of the boxes and one of you will select, without looking, one piece of paper from each box.

Choose a student to make a selection. Ask the student to read what is written on each piece of paper immediately after it is selected and then to return it to the box. Progressively record the results on the board as a number story. For example:

$$[illegible] + [illegible] = 2$$

T: The result is a number story. Is it true or false?

S: False. [illegible]

On the board indicate that the number story is false.

$$[illegible] + [illegible] = 2 \quad F$$

P7

Make two more trials of the experiment. For each trial, write the resulting number story on the board and record whether it is true or false. For example, you might get:

$\frac{3}{4} \times \frac{4}{3} < \frac{3}{4}$ **F** (since $\frac{3}{4} \times \frac{4}{3} = \frac{3 \times 4}{4 \times 3} = \frac{12}{12} = 1 > \frac{3}{4}$)

$\frac{2}{3} + \frac{1}{4} < 1$ **T** (since $\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12} < 1$)

Choose a student to perform another trial of the experiment. Proceed just as before, except stop the student before a paper is selected from the fifth box. Perhaps you will have this expression on the board:

$$\frac{1}{2} \times \frac{3}{2} <$$

T: Before you select from the fifth box, can we predict whether the resulting number story will be true or false?

S: It would be true if 1, 2, or $\frac{3}{2}$ were selected, since $\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$ and $\frac{3}{4}$ is smaller than 1, 2, and $\frac{3}{2}$. Otherwise, it would be false.

T: Which is more likely?

S: True, since there are three choices for a true number story and two choices for a false number story.

T: What is the probability that we'll get a true number story?

S: $\frac{3}{5}$ or 3 out of 5.

Instruct the student to select one from the fifth box and record the result on the board. For example:

$$\frac{1}{6} \blacksquare \frac{3}{6} < \frac{1}{3} \qquad F$$

Repeat this version of the experiment two or three times.

Exercise 2

Draw this table on the board near the five strings. In each location put the operation for a number story.

True	False
◆	◆
■	■

1. Using the numbers and symbols from these five strings, find a true number story using '<' and find a true number story using '>'. Also find a false number story using '<' and a false number story using '>'. Copy this table and put your four number stories in it.

Assist any students who need help. Encourage students to try to use both '<' and '>' in their examples. After several minutes of individual work, invite

P1

[illegible] for example.

True	False
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

Ask students to explain their selections. For example:

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

True

[illegible] + [illegible] < ☐

[illegible] + [illegible] < ☐

[illegible] + [illegible] < ☐

[illegible] + [illegible] = ☐

[illegible] [illegible]

[illegible]

True	Solutions
[illegible] + [illegible] < ☐	[illegible]
[illegible] + [illegible] < ☐	[illegible]
[illegible] + [illegible] = ☐	[illegible]
[illegible] + [illegible] = ☐	none

[illegible] [illegible]

P1

[illegible]: I think there are more three number stories.

[illegible]: How could we determine the probability of selecting a true number story?

[illegible]: By finding all the possible number stories and counting how many of them are true.

[illegible]: Next week we'll finish solving this problem.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

P 6

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

P 8

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

P 6

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]
[illegible]			
[illegible]			
[illegible]			

[illegible]	[illegible]	[illegible]	[illegible]
[illegible]			
[illegible]			
[illegible]			

[illegible]

[illegible]

[illegible]

Do not insist on the steps shown below; accept any correct methods.

$$\frac{3}{4} \times \frac{3}{2} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8} = 1\frac{1}{8} \qquad \frac{3}{4} + \frac{3}{2} = \frac{3}{4} + \frac{6}{4} = \frac{9}{4} = 2\frac{1}{4}$$

Compare the methods of multiplying and adding fractions. Then ask students to record the results in the tables.

$\times$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$\frac{1}{2}$			
$\frac{2}{3}$			
$\frac{3}{4}$			$1\frac{1}{8}$

$+$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$\frac{1}{2}$			
$\frac{2}{3}$			
$\frac{3}{4}$			$2\frac{1}{4}$

T: How will filling in these two tables help us to count how many true number stories there are?

S: In these tables, we will record all of the possible sums and products of the numbers from the first and the third strings. Then we can compare these sums and products to the numbers in the fifth string to count how many true number stories we could get.

T: Complete both tables on Worksheet P85 on your own.

Allow sufficient time for this task. Assist any students needing help. As the students are working, invite students one at a time to write the correct entries in the tables on the board.

$\times$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{2}{3}$	$\frac{1}{6}$	$\frac{8}{9}$	1
$\frac{3}{4}$	$\frac{3}{16}$	1	$1\frac{1}{8}$

$+$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{3}{4}$	$1\frac{5}{6}$	2
$\frac{2}{3}$	$\frac{11}{12}$	2	$2\frac{1}{6}$
$\frac{3}{4}$	1	$2\frac{1}{12}$	$2\frac{1}{4}$

T: How could we get a true number story using "="?

S: Select $\frac{3}{4}$ from the fifth string to get "$\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$".

T: Is there another way?

S: Select 2 from the fifth string to get "$\frac{2}{3} + \frac{4}{3} = 2$".

T: Who can explain how to use the tables to find true number stories using "="?

S: Compare the numbers in the tables to the numbers in the fifth string.

T: How many true number stories are there using "="?

Give the students a few minutes to determine this.

S: Seven, because "2" appears in the tables twice, "1" appears three times, and "$\frac{3}{4}$" appears twice. "$\frac{1}{3}$" and "$\frac{5}{2}$" do not appear.

Write this information about true number stories using "=" on the board.

$$= \qquad 7T$$

T: There are seven true number stories using "=". How can we count the true number stories using "<"?

Let the class discuss this question. Lead them to the idea that each number in the fifth string can be considered individually.

T: Using the two tables on the board, can you tell me a true number story that ends in "< 2"?

S: "$\frac{2}{3} \times \frac{4}{3} < 2$" is true, because $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$ and $\frac{8}{9} < 2$.

S: "$\frac{1}{2} + \frac{4}{3} < 2$" is true, because $\frac{1}{2} + \frac{4}{3} = 1\frac{5}{6} < 2$.

T: How many true number stories end with "< 2"?

S: Thirteen, because thirteen of the eighteen entries in the two tables are less than 2.

Record this information on the board.

$$\begin{array}{ll} = & 7T \\ <2 & 13T \end{array}$$

T: How many true number stories end with "< 1"?

S: Eight, because eight entries in the tables are less than 1.

Record this information on the board.

=	7 T
<2	13 T
<1	8 T

T: How many true number stories end with "$<\frac{5}{2}$"?

S: Eighteen, because all the entries in the two tables are less than $\frac{5}{2}$ ($2\frac{1}{2}$) and there are eighteen entries, nine in each table.

=	7 T	$<\frac{5}{2}$	18 T
<2	13 T		
<1	8 T		

T: Are there any true number stories that end in "$<\frac{1}{3}$"?

S: Yes, "$\frac{1}{2} \times \frac{1}{4} < \frac{1}{3}$" is true because $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ and $\frac{1}{8} < \frac{1}{3}$.

S: "$\frac{2}{3} \times \frac{1}{4} < \frac{1}{3}$" is true because $\frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$ and $\frac{1}{6} < \frac{1}{3}$.

S: "$\frac{3}{4} \times \frac{1}{4} < \frac{1}{3}$" is true because $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16} < \frac{1}{3}$.

T: Why is $\frac{3}{16} < \frac{1}{3}$?

S: $\frac{5}{16}$ is about $\frac{1}{3}$, so $\frac{3}{16} < \frac{1}{3}$.

S: $\frac{1}{3} = \frac{3}{9}$ and $\frac{3}{16} < \frac{3}{9}$ because $\frac{1}{16} < \frac{1}{9}$, so $\frac{3}{16} < \frac{1}{3}$.

T: How many true number stories end with "$< \frac{1}{3}$"?

S: Three.

Record this information.

$=$	7 T	$< \frac{5}{2}$	18 T
< 2	13 T	$< \frac{1}{3}$	3 T
< 1	8 T		

T: We still must find how many true number stories end in "$< \frac{3}{4}$". Write on your paper all the entries in the two tables that are smaller than $\frac{3}{4}$.

Give the students a few minutes to do this.

T: Which numbers are smaller than $\frac{3}{4}$?

S: $\frac{1}{8}$, $\frac{1}{6}$, $\frac{3}{16}$, and $\frac{2}{3}$.

T: Therefore, how many true number stories end with "$< \frac{3}{4}$"?

S: Four.

Record this information.

$=$	7 T	$< \frac{5}{2}$	18 T
< 2	13 T	$< \frac{1}{3}$	3 T
< 1	8 T	$< \frac{3}{4}$	4 T

P 8

T: Have we counted all the true number stories?

S: Yes.

T: How many different true number stories could we get from a selection from the five boxes (strings)?

S: Fifty-three; 7 + 13 + 8 + 18 + 3 + 4 = 53.

T: How many possible number stories are there?

S: One hundred eighty.

T: What is the probability of getting a true number story from a selection from the five boxes?

S: 53 out of 180 or $\frac{53}{180}$.

Write "$\frac{53}{180}$" on the board.

T: What simple fraction is close to $\frac{53}{180}$?

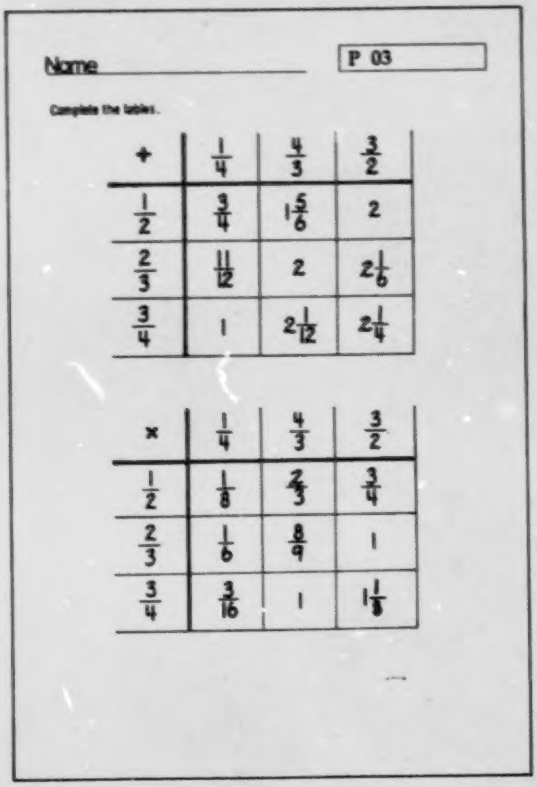
S: $\frac{1}{3}$.

Write "$\frac{1}{3} = \frac{\square}{180}$" on the board.

T: $\frac{1}{3}$ is how many one hundred eightieths?

S: 60.

T: Yes, $\frac{1}{3} = \frac{60}{180}$; therefore, $\frac{53}{180}$ is a little less than $\frac{1}{3}$.



Name ______________________

P 03

Complete the tables.

$+$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{3}{4}$	$1\frac{5}{6}$	2
$\frac{2}{3}$	$\frac{11}{12}$	2	$2\frac{1}{6}$
$\frac{3}{4}$	1	$2\frac{1}{12}$	$2\frac{1}{4}$

$\times$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{2}{3}$	$\frac{1}{6}$	$\frac{8}{9}$	1
$\frac{3}{4}$	$\frac{3}{16}$	1	$1\frac{1}{8}$

2

PROBABILITY STORIES

Four stories ranging from predicting the future to choosing volleyball teams provide the settings of the probability questions in this chapter. The predominant method is to divide a square to reflect the choices in a situation and to color the sections according to the possible outcomes; for example, the outcome may be success or failure or it may be which team you are on. The area in each color compared to the area of the whole square gives the probability for the corresponding outcome. This case-by-case approach to finding a solution can be used in a variety of mathematical settings, probability situations being only one of them.

CAPSULE LESSON SUMMARIES

Section 2.1 :	Tell a story about a princess who can help the one she loves in a situation which could mean his death or their marriage. Find the probabilities involved in a choice which confronts her. (Worksheets P04, P05)
Section 2.2 :	Bruce suggests a method of choosing two teams. Decide whether the method is fair or if Bruce can use it to his advantage.
Section 2.3	Tell a story about Sylvia who is invited to a beach party on an island. A problem arises since the last ferry leaves in 60 minutes which is not enough time for Sylvia to take her usual route. Find the probability that Sylvia will get to the ferry on time if she follows paths through an unfamiliar swamp. (Worksheets P06, P07, P08)
Section 2.4 :	Calculate the probability of forming one long piece of rope when some of the ends of three ropes are joined by two knots. Vary how the ropes are held and analyze the new situation which results. (Worksheets P09, P10)

P 2

2.1 Marriage by Chance

CAPSULE LESSON SUMMARY

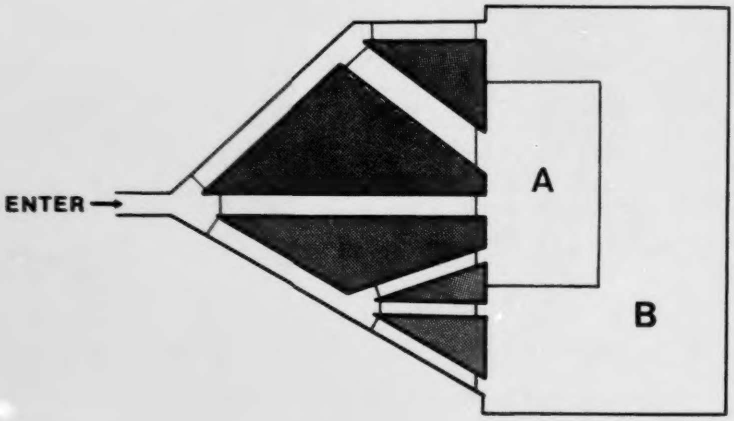
Tell a story about a princess who can help the one she loves in a situation which could mean his death or their marriage. Find the probabilities involved in a choice which confronts her.

MATERIALS

Teacher: Colored chalk; Probability Poster #1 for IG-V
Student: Colored pencils; paper; ruler; Worksheets P04 and P05

DESCRIPTION OF LESSON

Display Probability Poster #1.



Tell the following story in a manner appealing to your class. Stop periodically to let your students discuss the story.

T: I have a love story to tell you about a beautiful princess. The king and queen arranged that their daughter, the princess, would marry Prince Cuthbert from a nearby kingdom. The princess did not consider Cuthbert very appealing, but it was the royal tradition that she had to accept this marriage planned by her parents. The princess must follow tradition.

T: The wedding day approached and there were no problems until the princess met Reynaldo in the village. Reynaldo was handsome, clever, and romantic. It was love at first sight. But being a poor farmer, Reynaldo was not even allowed to talk to the princess. Despite the laws, they began to see each other secretly. Reynaldo and the princess were careful, but one day the king's spies caught them together. The king was extremely angry that Reynaldo, a commoner, would dare to fall in love with his daughter.

What do you think was the punishment for this?

Let the students suggest possible punishments.

T: In this country, the punishment was to put the person in a room full of tigers. So the King ordered that Reynaldo be sent to the tigers. But the princess wept and protested that she now loved Reynaldo and would never marry Cuthbert. The king was very upset and confused. According to tradition, Reynaldo should be punished. The king would lose face if he allowed the princess to marry Reynaldo instead of Cuthbert. But still the king loved his daughter and did not wish to hurt her. Because of his daughter's plea, the king agreed to discuss the problem with the queen. For the night, he sent Reynaldo to the dungeon instead of giving him to the tigers.

What should the king do?

Encourage the students to comment on the king's dilemma.

T: The next morning the king called for Reynaldo and the princess to tell them of his plan. "The queen and I cannot decide whether to let you, Reynaldo, marry our daughter or to send you to the tigers. Instead we will let the two of you determine your own fate." The king continued, "Tonight, Reynaldo will walk through a maze and arrive at a door to one of two rooms. He must open the door and enter the room."

P 2

Tell your class that the poster is a picture of the maze, but that Reynaldo, of course, will not see such a picture. Ask your class what they think will be waiting in each room.

T: The king told Reynaldo that the princess would be waiting in one room and there would be tigers in the other room. He then sent Reynaldo back to the dungeon to wait until it was time for him to walk the maze. He turned to his daughter, "If you love Reynaldo and are wise, you can help him. Here is a map of the maze." The king showed the princess a picture like this one (point to the poster). He then explained to her, "Reynaldo has not seen this maze. He must enter, choose a door, and walk down one of the three paths. Each door is made so that it springs open even if he opens it only slightly. Also each door closes behind him so he cannot turn back. He must continue until he enters one of the rooms and finds either you or the tigers. The rooms are soundproof, so Reynaldo will not be able to hear you or the tigers. If Reynaldo finds you, I will allow you to marry him. Otherwise, you will marry Cuthbert."

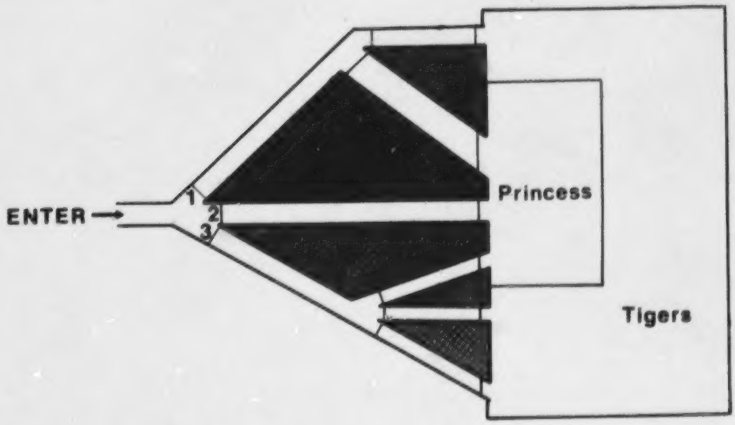
The princess agreed to this plan, because she could choose which room to wait in, A or B. The tigers would be put in the other room.

Discuss the princess' choice with the class. You might ask the following questions.

T: Reynaldo, of course, can only guess which paths to follow. But do you think that he is more likely to enter room A or room B? Or are the chances for each the same? If you were the princess, which room would you choose to stay in?

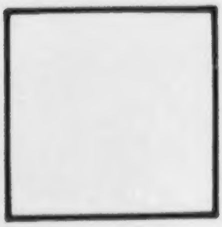
Encourage discussion. Some students might suggest that Reynaldo's probability for entering each room is $\frac{3}{6}$ or $\frac{1}{2}$ because there are three doors into each room. If no one questions this response, you should ask whether Reynaldo is equally likely to arrive at each of the six doors. The class should realize that Reynaldo is more likely to arrive at some doors (e.g., the third door from the top) than at other doors (e.g., the fifth door from the top) due to the layout of the maze. Therefore, we cannot easily tell if Reynaldo's probability of entering each room is $\frac{1}{2}$.

After a class discussion of Reynaldo's choice, take a vote on which room the princess should choose. Then mark each room "Princess" or "Tigers" as decided by the vote. Number the paths near the entrance "1", "2", and "3". The dialogue in the remainder of this lesson is based on placing the princess in room A. If your class decides to put the princess in room B, use the same dialogue and pictures, but reverse all references to the princess and to the tigers.



T: Now we will determine if you have made the best choice for the princess.

Draw a square on the board.

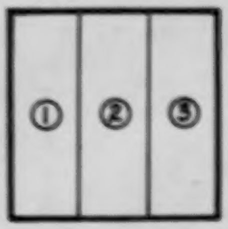


P2

T: We will use this square to show what could happen to Reynaldo. When he enters the maze, he must choose one of the three paths to follow. How can I indicate this choice on the square?

S: Divide the square into three parts of the same size since each of the three paths is equally likely to be choosen by Reynaldo.

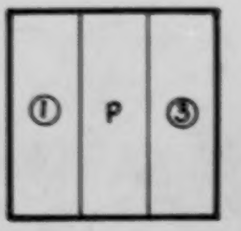
Carefully divide the square into thirds and label the pieces.



T: Suppose Reynaldo chooses path 2. What happens?

S: He would go straight to the room with the princess.

T: I will mark the region for path 2 with a "P".



T: Now, suppose that he chooses path 1. What happens?

S: His chances of going to the princess are the same as his chances of going to the tigers.

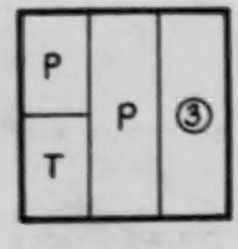
T: Why?

S: After following path 1 he comes to a junction and must select one of two paths. One path leads to the blue room and one path leads to the red room.

T: How can I show that on the square?

S: Divide the region for path 1 into two parts of the same size. Mark one part "P" for the princess and one part "T" for the tigers.

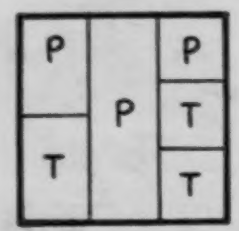
Accurately divide the region for path 1 in half.



T: Let's see what happens if Reynaldo chooses path 3.

S: He would come to a junction with three doors. Reynaldo would have two chances to go to the tigers and one chance to go to the princess. On the square, divide the region for path 3 into three parts of the same size. Mark two parts with a "T" and one part with a "P".

Accurately divide the region for path 3 into thirds and mark the parts appropriately.



T: Is Reynaldo more likely to find the room with the princess or the room with the tigers?

P 2

The students are likely to observe that more of the square is marked "P" than is marked "T".

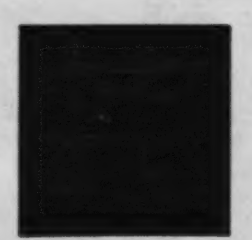
T: Look at the paths in the maze. Can anyone explain why Reynaldo is more likely to find the room with the princess than the room with the tigers?

S: Reynaldo is very lucky if he chooses path 2 which leads directly to the princess. If he chooses path 1 or path 3, he could go to the princess or to the tigers.

Color the regions of the square marked "P" in blue and color the regions labeled "T" in red.

T: I colored the regions for the princess in blue and the regions for the tigers in red. Let's calculate exactly Reynaldo's chances of finding the princess and his chances of finding the tigers. Can anyone divide the square into small pieces of the same size so we can count the number of pieces for red and for blue?

Invite several students to the board to explain methods of dividing the square. Several methods are possible. The most natural division is shown below.



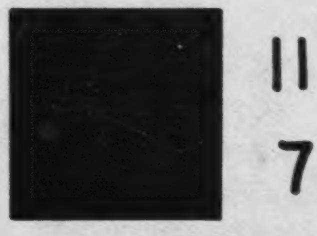
T: How many pieces are colored blue?

S: Eleven.

T: How many pieces are colored red?

S: Seven.

Near the square write "11" in blue and "7" in red.



T: Reynaldo has eleven chances to find the princess and seven chances to find the tigers. We can say also that Reynaldo's chances are 11 out of 18 to find the princess and 7 out of 18 to find the tigers.

Write "$\frac{11}{18}$" in blue and "$\frac{7}{18}$" in red near the square.

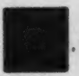
T: What would the results have been if you had placed the princess in the other room?

S: Just the opposite, Reynaldo would have eleven chances to find the tigers and seven chances to find the princess.

Many students are likely to be curious how the story ends, that is, what "really" happened to Reynaldo when he walked the maze. We suggest that you invent an appropriate happy ending to the story.

Worksheets P04 and P05 are available. After many students have attempted P05, solve the problem collectively if time permits.

Name ______________________ **P 04** *

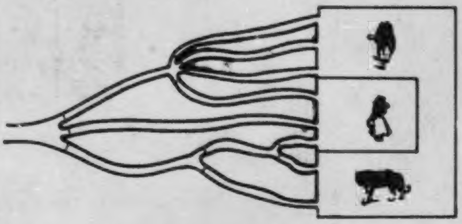
How many squares of this size, , fit into the red region? 13

How many squares of this size, , fit into the blue region? 12

Use a ruler if you wish.

Name ______________________ **P 05** **

The King has another maze near the castle. If Reynaldo goes through this maze, find his probability of entering the room with the princess.



Use this square to help you solve the problem.

What are Reynaldo's chances of finding the princess? $\frac{6}{12}$ or $\frac{1}{2}$

What are Reynaldo's chances of finding the tigers? $\frac{6}{12}$ or $\frac{1}{2}$

2.2 Who's on Bruce's Team?

CAPSULE LESSON SUMMARY

Bruce suggests a method of choosing two teams. Decide whether the method is fair or if Bruce can use it to his advantage.

MATERIALS

Teacher: Three red and three blue marbles; two identical small boxes; colored chalk

Student: Colored pencils; paper

DESCRIPTION OF LESSON

Tell the following story to your class stopping occasionally to let students comment on the story. Your students might remember Bruce from probability situations in IG-II. Bruce also appeared (with Rita) in lessons in IG-IV. Have three red marbles, three blue marbles, and two boxes on hand.

T: Bruce is now a sixth-grader. One day his class is going to the gym to play volleyball. The teacher suggests that they choose two teams before going to the gym, but the students start arguing loudly about how to choose the teams fairly.

Do you know any fair methods to select teams?

Let the students briefly suggest various methods. Then as you continue the story, show the class the six marbles and the two boxes.

T: Bruce suggests a way to pick teams. He tells the class, "I have three red marbles and three blue marbles. Our teacher selects one of you as a candidate for my team. Then I put the six marbles into these two boxes, placing at least one marble in each box. I mix up the boxes and the candidate selects a box and then, without looking, picks a marble. That person is on my team if the marble is blue and on the other team if the marble is red. We keep doing this until half the class is on one team."

P 3

Write the following information on the board.

● Bruce's team
● Other team

T: Let's try Bruce's method. Who wants to be Bruce?

Invite a student to play the role of Bruce. Demonstrate Bruce's method by following these steps.

1. Choose a student as the first candidate for "Bruce's" volleyball team.
2. Bruce puts the six marbles into the two boxes, with at least one marble in each box.
3. Bruce closes the boxes and shuffles them behind his back.
4. The candidate chooses one of the boxes.
5. The candidate selects, without looking, one marble from the box.
6. Determine if the student is on "Bruce's" team (by choosing a blue marble) or on the other team (by choosing a red marble).

Repeat this procedure several times. Each time select another student as a candidate for "Bruce's" team and then let "Bruce" change the distribution of marbles, if he wishes. Continue until the class understands Bruce's method of choosing teams.

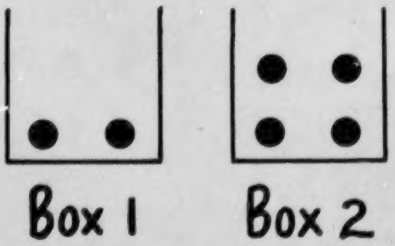
T: Do you think Bruce's method is fair or do you think that he could use it to get many of the better players on his team and the weaker players on the other team?

Let the students express their opinions. Be sure that your students understand that all six marbles are used each time and that Bruce puts the marbles in the boxes after the teacher has selected a candidate.

T: Do you think Bruce's distribution of the six marbles is important? For example, if Bruce wants a certain good volleyball player on his team, could he put the marbles in the boxes in a certain way to make it likely that this classmate will select a blue marble?

Lead a class discussion of these questions without reaching any definite conclusions.

On the board draw the following picture of the marbles and the boxes.



T: If Bruce places the marbles as I have indicated, do you think the probability of the respective candidate being on Bruce's team is $\frac{1}{2}$, larger than $\frac{1}{2}$, or smaller than $\frac{1}{2}$?

Let the students express their opinions, but do not indicate the correct answer. Draw a square on the board.

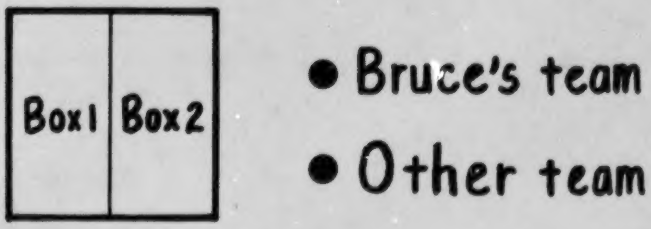
T: Let's use this square to calculate the classmates' probability of being on Bruce's team. We'll analyze the possible choices one-by-one and use the square to record the results of those choices. After Bruce distributes the marbles, closes the boxes, and mixes then behind his back, what does the classmate do first?

P 3

S: Selects one of the two boxes.

T: How can I show the choice on the square?

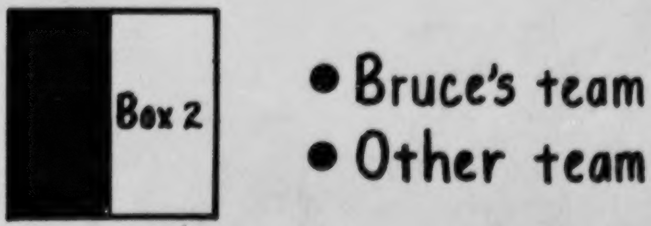
S: Divide the square in half since each box is equally likely to be chosen.


Box 1
Box 2
• Bruce's team
• Other team

T: What happens if the classmate selects Box 1?

S: The classmate gets a red marble for sure and is not on Bruce's team.

T: I'll color half the square red to show this.


Box 2
• Bruce's team
• Other team

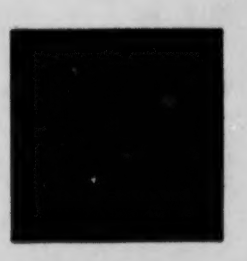
T: What happens if the classmate chooses Box 2?

S: The chances are 3 out of 4 that the classmate will be on Bruce's team.

T: How can I show that on the square?

S: Divide the region for Box 2 into four parts of the same size. Color one part red and three parts blue.

Invite a student to divide the region and to do the coloring; for example,



- ● Bruce's team
- ● Other team

Encourage the class to discuss this result for the given distribution and then ask some questions about it.

T: If Bruce divides the marbles in this way, is his classmate more likely to be on Bruce's team or on the other team?

S: The other team, because more than half of the square is red.

T: What are the classmate's chances of being on Bruce's team? ... on the other team?

S: Three chances to be on Bruce's team and five chances to be on the other team.

Write "3" in blue and "5" in red near the square. Point to the small red part of the square and note that three pieces of that size fit into the blue region and five fit into the red region.

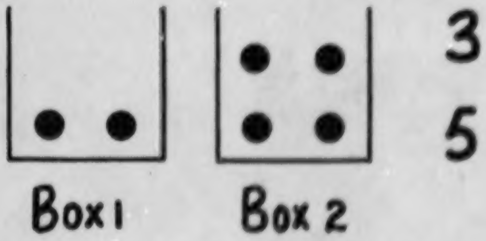
3

5

- ● Bruce's team
- ● Other team

P 3

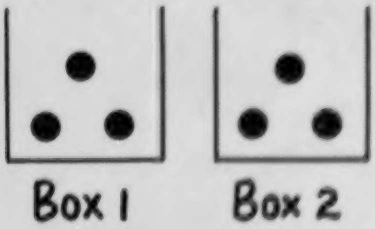
Record the chances near the marble distribution at one end of the board.



T: For this distribution, what is the probability of being on Bruce's team?

S: $\frac{3}{8}$ or 3 out of 8.

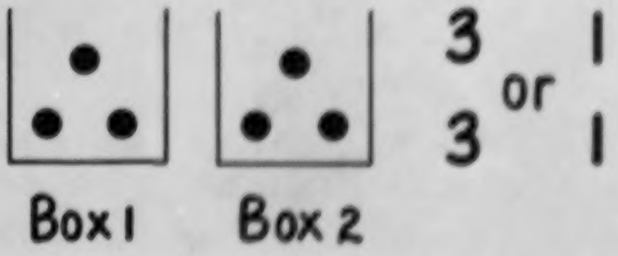
Invite a student to suggest a different distribution of the six marbles in the two boxes and to draw a picture to represent the distribution; for example,



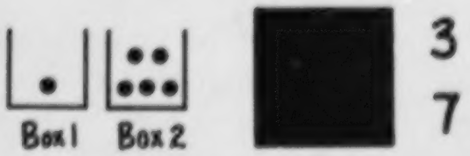
T: Draw a square on your paper and use it to calculate a classmate's chances of being on Bruce's team for this distribution.

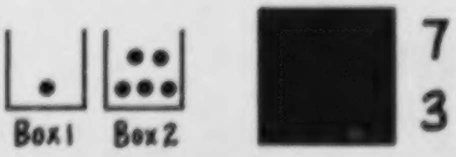
Assist students who need help getting started. After about five minutes, solve the problem as a class at the board.

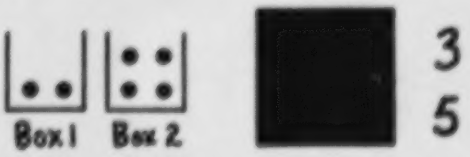
Record on the board the result of the problem solved in your class; for example,

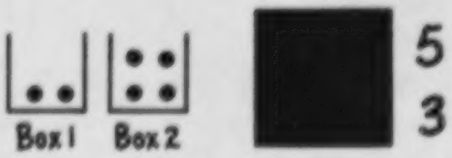


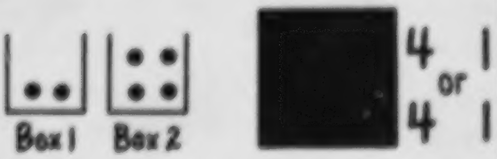
For your reference, the solutions for the seven distinct distributions are shown below.

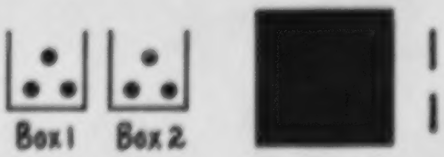

Box 1
Box 2
3
7

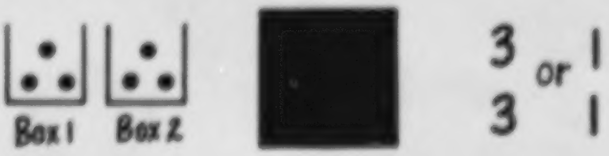

Box 1
Box 2
7
3


Box 1
Box 2
3
5


Box 1
Box 2
5
3


Box 1
Box 2
4 or 1
4 1


Box 1
Box 2
1
1

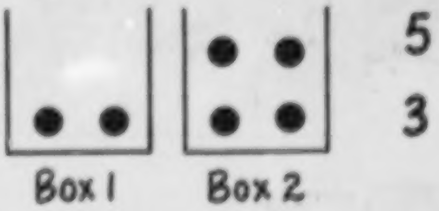

Box 1
Box 2
3 or 1
3 1

By comparing the solutions for the two distributions studied in your class, point out that changing the distribution of the marbles could change a classmate's probability of being on Bruce's team.

T: Pretend that you are Bruce and that the next classmate to choose a marble is a good volleyball player whom you want on your team. How would you distribute the six marbles into the two boxes to have the largest probability of getting this classmate on your team? To find out, select the distribution you feel is best. Using a square, calculate the classmate's chances of being on Bruce's team. Then try other distributions until you are convinced you have found the best distribution.

P 3

As the students work, help any students needing assistance. After about ten minutes, ask students to tell which distributions they chose and the results for those distributions. Record each solution by drawing a picture on the board to represent it; for example,

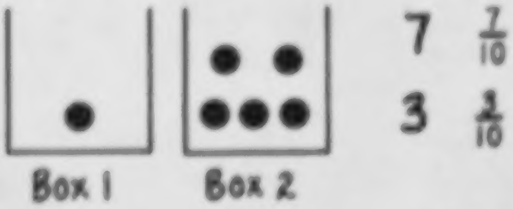


Use the answer key provided earlier to check solutions (see page 53). If there is disagreement over a particular answer, draw a square and find the chances collectively.

T: How could the marbles be distributed so that the classmate will have the largest probability of being on Bruce's team?

S: Put one blue marble in one box and put the other five marbles in the other box. For this distribution, the classmate's probability of being on Bruce's team is $\frac{7}{10}$ or 7 out of 10.

Draw the corresponding picture on the board.



Draw a square on the board and invite a student to explain how to find the chances for this distribution. A picture similar to the following should result.

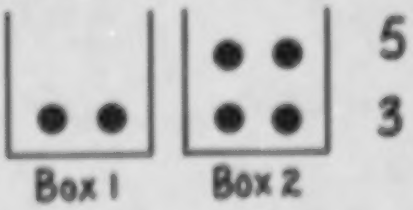
T: Did anyone find a distribution of the six marbles which results in more of the square being colored blue?

S: No.

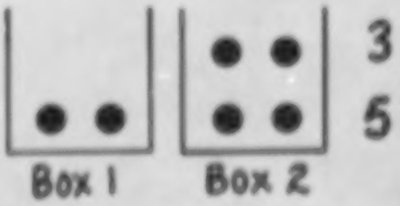
T: What other distributions did some of you try?

Record other distributions and the corresponding chances on the board. Use a square to check a student's distribution only if the results are incorrect or if there is disagreement over it.

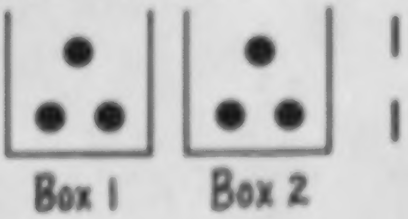
Invite the class to compare and discuss the results recorded on the board. Students might notice the symmetry involved when the red and the blue marbles are switched; for example,



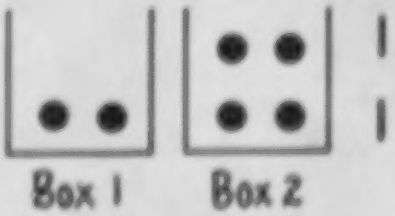
and



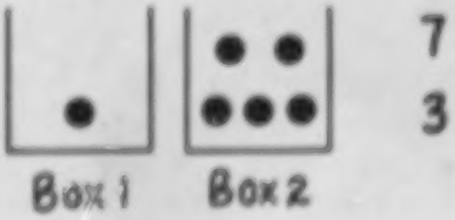
Also, students might notice that they can predict that the probability of being on Bruce's team is $\frac{1}{2}$ simply by studying the distribution, without dividing a square. For example,



or



Refer to the following picture on the board.



P 3

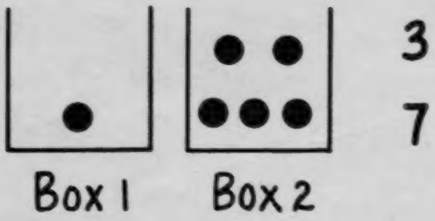
T: Can you explain why this distribution gives the largest probability of being on Bruce's team?

S: It only takes one blue marble to insure complete success with one of the boxes. Then we put two blue marbles in with the red marbles in the other box; the chances of getting a blue marble from that box is close to 1 out of 2.

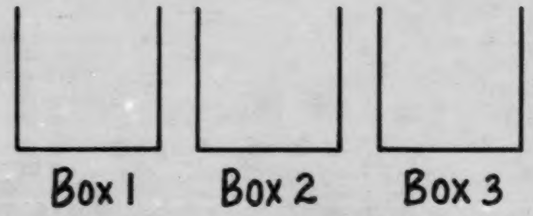
T: If there is a person in Bruce's class whom Bruce does not want on his team, how should he distribute the marbles for that person?

S: Put one red marble in one box and the other five marbles in the other box. Then the probability of that person being on Bruce's team is only $\frac{3}{10}$ or 3 out of 10.

Draw a picture on the board to represent this distribution if it is not already recorded.



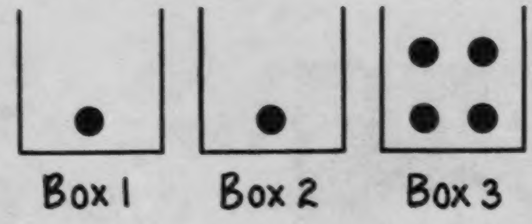
Draw the following picture on the board.



T: What if Bruce has three boxes instead of two and still has three red and three blue marbles. How should he distribute the marbles for a classmate if he wants the largest probability of getting that person on his team?

315

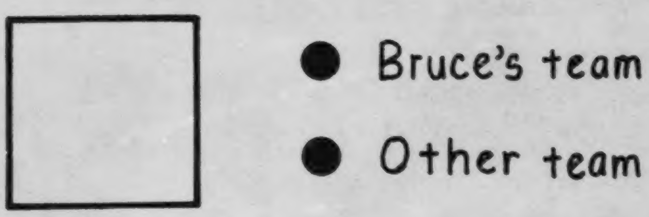
S: Put one blue marble in each of two boxes and put the other four marbles in the third box.


Box 1
Box 2
Box 3

T: Do you think using three boxes instead of two boxes increases or decreases the chances for Bruce to get a friend on his team?

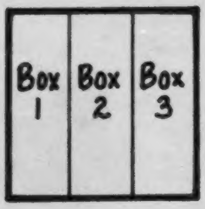
S: Increases.

Draw a square on the board.


● Bruce's team
● Other team

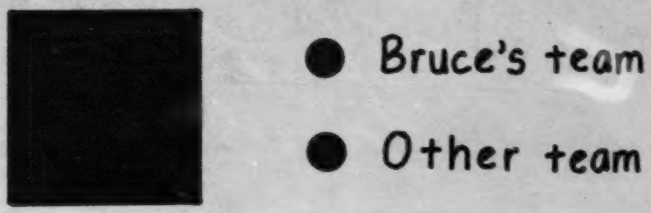
T: How should I divide this square to calculate the probability that a classmate choosing from this distribution will be on Bruce's team?

S: First, divide the square into thirds because there are three boxes.


Box 1
Box 2
Box 3

P3

S: Color the regions for Box 1 and Box 2 blue. Then divide the region for Box 3 into fourths and color one piece blue and three pieces red.



T: What are the chances that Bruce's classmate will be on Bruce's team? . . . the other team?

S: Nine chances to be on Bruce's team and three chances to be on the other team.

Write "9" in blue and "3" in red near the square.

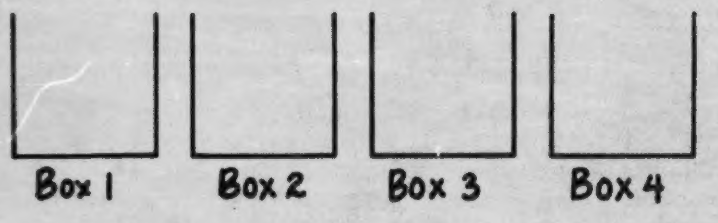
T: What is the classmate's probability of being on Bruce's team?

S: 9 out of 12 or $\frac{9}{12}$, or $\frac{3}{4}$ since $\frac{9}{12} = \frac{3}{4}$.

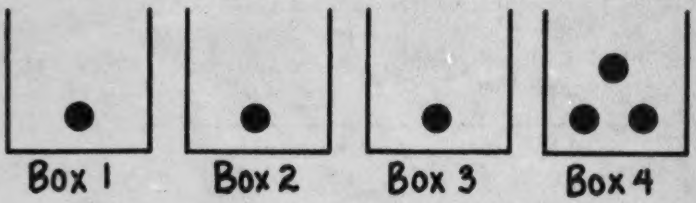
Refer to the results "9-3" for three boxes and "7-3" for two boxes. Ask the class which result represents a higher probability of being on Bruce's team. Let the students compare the probabilities as they wish; for example,

- 9-3 is better than 7-3 since in each there are three chances to be on the other team, but there are more chances to be on Bruce's team for 9-3 than 7-3;
- $\frac{9}{12} = \frac{3}{4} = 0.75$, while $\frac{7}{10} = 0.7$ and $0.75 > 0.7$;
- $\frac{9}{12} = \frac{3}{4}$ and $\frac{3}{4} > \frac{7}{10}$;
- $\frac{9}{12} = \frac{3}{4} = \frac{15}{20}$ and $\frac{7}{10} = \frac{14}{20}$; therefore, $\frac{3}{4} = \frac{15}{20} > \frac{14}{20} = \frac{7}{10}$.

Draw the following picture on the board.



T: If there are four boxes and the same six marbles, what is the best way to distribute the marbles if Bruce wants a person to be on his team?



T: What is the probability of being on Bruce's team for this distribution?

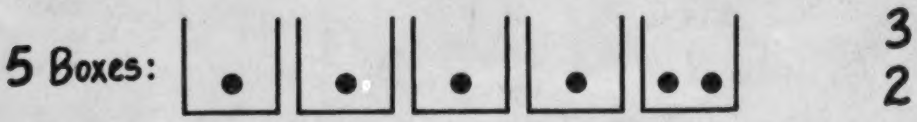
S: $\frac{3}{4}$ or 3 out of 4.

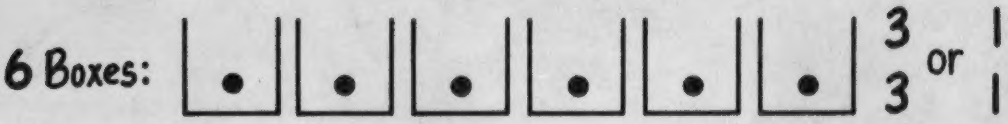
T: For Bruce to get a classmate on his team, is using four boxes better than three boxes?

S: No, the largest probability for each situation is $\frac{3}{4}$.

If they wish, let the class discuss why using three boxes instead of two boxes increases the probability of being on Bruce's team, but using four boxes does not further increase the probability.

If there is time, present the same situation using five boxes and then using six boxes and notice that the probabilities decrease with the addition of more boxes.





P 6

2.3 Time is of Importance

CAPSULE LESSON SUMMARY

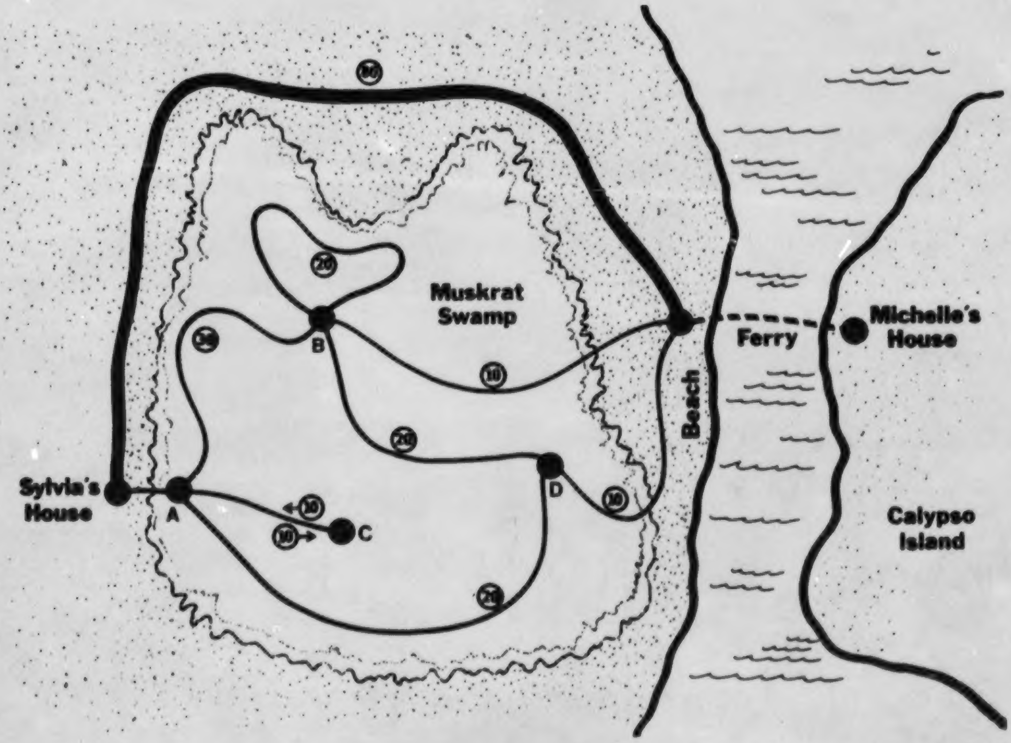
Tell a story about Sylvia who is invited to a beach party on an island. A problem arises since the last ferry leaves in 60 minutes which is not enough time for Sylvia to take her usual route. Find the probability that Sylvia will get to the ferry in time if she follows paths through an unfamiliar swamp.

MATERIALS

Teacher:	Probability Poster #2 for IG-V; colored felt-tip pens; meter stick; colored chalk
Student:	Colored pencils; paper; ruler; Worksheets P06, P07, and P08

DESCRIPTION OF LESSON

Display Probability Poster #2.


20
Muskrat
Swamp
B
10
Michelle's
House
Ferry
Beach
20
D
Sylvia's
House
A
←10
10→
C
10
Calypso
Island
20

Refer to the poster as you tell the following story about Sylvia's vacation near the coast. Make the story as interesting as possible, stopping occasionally to let the students discuss the situation.

T: Sylvia is a lucky young girl; she spends one month every summer with her aunt and uncle near the ocean. Sylvia often rides her bicycle to the beach and she always rides on the main road around Muskrat Swamp. (Trace the heavy black path from "Sylvia's House" to the "Beach".) It takes her 80 minutes if she does not stop. There are some paths through Muskrat Swamp on which Sylvia could ride on her bicycle, but she doesn't know which paths to take and is a little afraid of getting lost in the dark swamp.

Sylvia's friend, Michelle, lives on Calypso Island. To visit Michelle, Sylvia must take a ferry which runs only three times a day; morning, noon, and evening.

Let the students talk about the story; for example, they may have comments on the swamp, the ferry, the island, and so on.

T: Late one afternoon, Michelle calls Sylvia on the telephone and invites her to a party that evening on Calypso Island. They would swim, they would build a bonfire on the beach, and Sylvia could stay overnight. Sylvia is excited, but realizes that it is almost 5:00 now and that the last ferry to Calypso Island leaves at 6:00. She tells Michelle, "It takes me 80 minutes to bicycle to the ferry so I would be too late. And my aunt and uncle are not here now." Michelle tells her that there are shortcuts through Muskrat Swamp. Sylvia is afraid of the swamp and of getting lost, but Michelle finally convinces her to try it.

So Sylvia jumps on her bicycle and begins her ride through the swamp. Michelle could not give her good directions and there are no signs, so she has to guess which paths to follow. The number on each path tells how many minutes it takes to ride a bicycle along it. Sylvia has exactly 1 hour (60 minutes). Do you think she will get to the ferry on time?

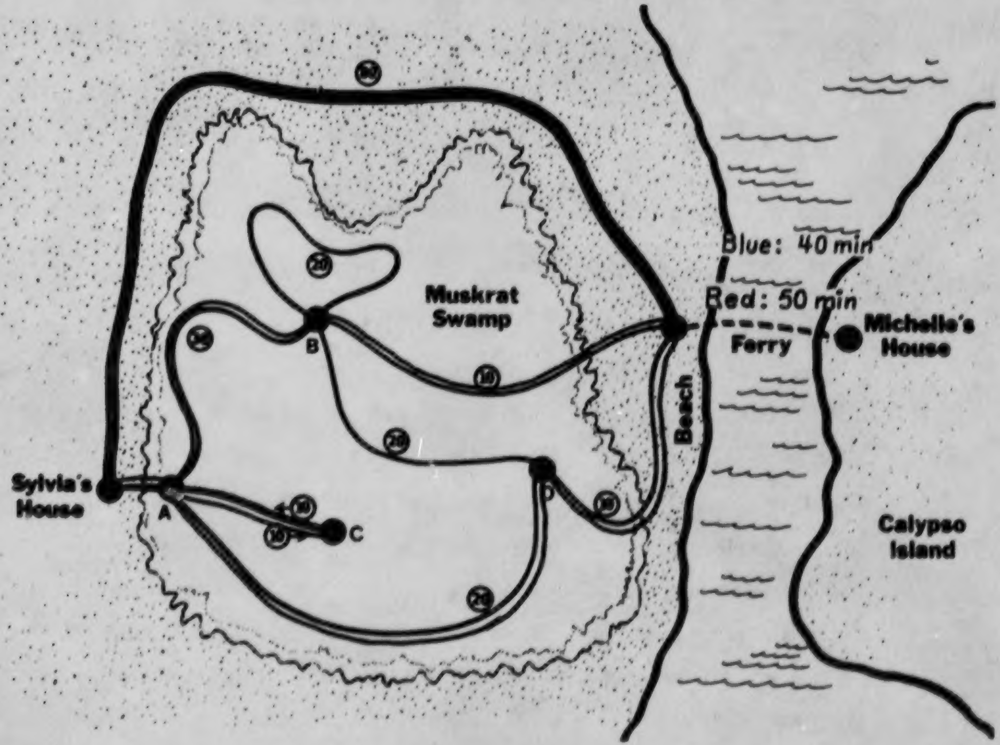
P 6

Let the students study the map and express their opinions. Be sure that the dead-end path from A to C and the loop at B are both pointed out.

Distribute copies of Worksheet P06.

T: Who can trace a path which Sylvia could take and get to the ferry in 60 minutes or less?

Invite about three students to trace paths which start at Sylvia's house, end at the ferry, and take at most 60 minutes. Two such paths are shown below.

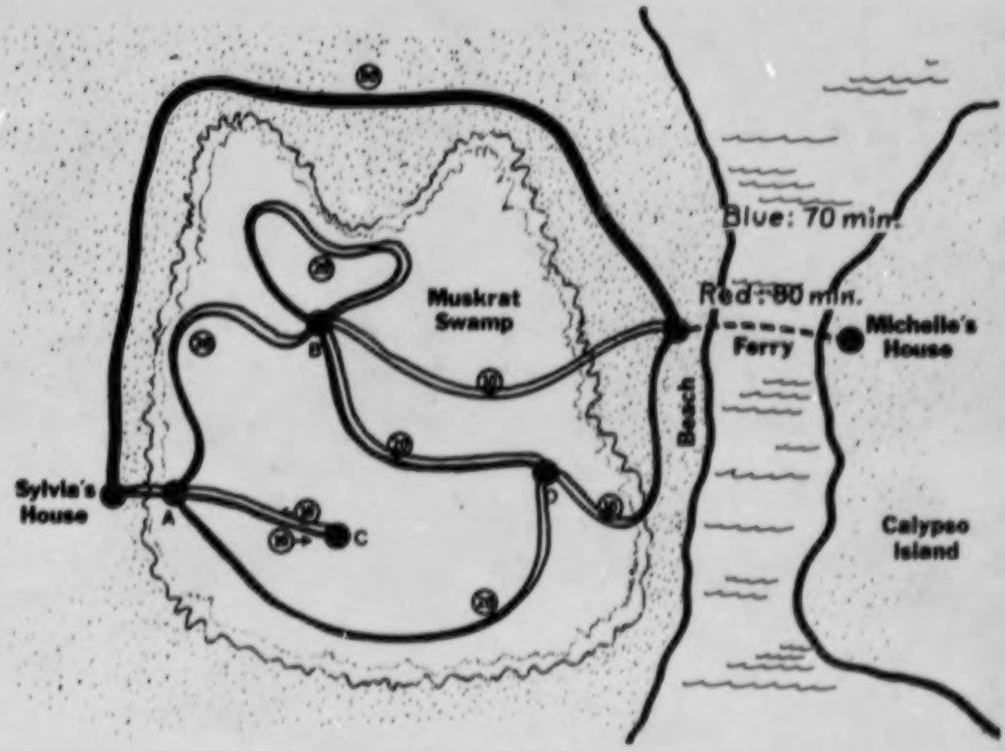

Muskrat Swamp
B
20
10
20
D
10
Sylvia's House
A
10
10
C
20
Blue: 40 min
Red: 50 min
Ferry
Michelle's House
Beach
Calypso Island

T: Do you think Sylvia would take the same path twice?

Lead to the idea that if Sylvia is clever and marks each path she takes, for example with cloth or a large stick, she could avoid taking the same path twice.

T: Who can trace a path which Sylvia could take and not get to the ferry in 60 minutes or less?

Invite two or three students to trace paths which start at Sylvia's house, end at the ferry, and take more than 60 minutes. Two such paths are shown below.


Blue: 70 min.
Red: 80 min.
Muskrat Swamp
Sylvia's House
A
B
C
D
Ferry
Michelle's House
Beach
Calypso Island

T: Sylvia does get to the ferry on time and joins Michelle's party on Calypso Island. When Sylvia arrives, she tells Michelle, "I was really lucky to get to the ferry on time because there are a lot of paths which I could have taken through the swamp." Michelle answers, "There's not much luck involved. It's easy to ride through Muskrat Swamp in 60 minutes or less."

What do you think? How lucky was Sylvia if she always had to guess which path to take?

P 6

Encourage students to estimate Sylvia's chances or probability of getting through Muskrat Swamp in 60 minutes and list their estimates on the board. Ask the students if they feel Sylvia's probability of arriving on time is very low, less than $\frac{1}{2}$, or very high, more than $\frac{3}{4}$.

Draw a large square on the board.

T: Let's use this square to calculate Sylvia's probability of riding through the swamp in 60 minutes or less. We must study all the paths Sylvia could take and all the choices she could make. We will represent these paths and choices on the square.

Sylvia enters the swamp here (point to "A" on the map). How many choices of paths does she have at A?

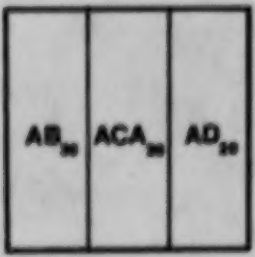
S: Three; she could take the path to B, to D, or to the dead-end at C and back to A.

T: How could I show on the square that Sylvia has three choices?

S: Divide the square into thirds because each of the three paths is equally likely to be chosen by Sylvia.

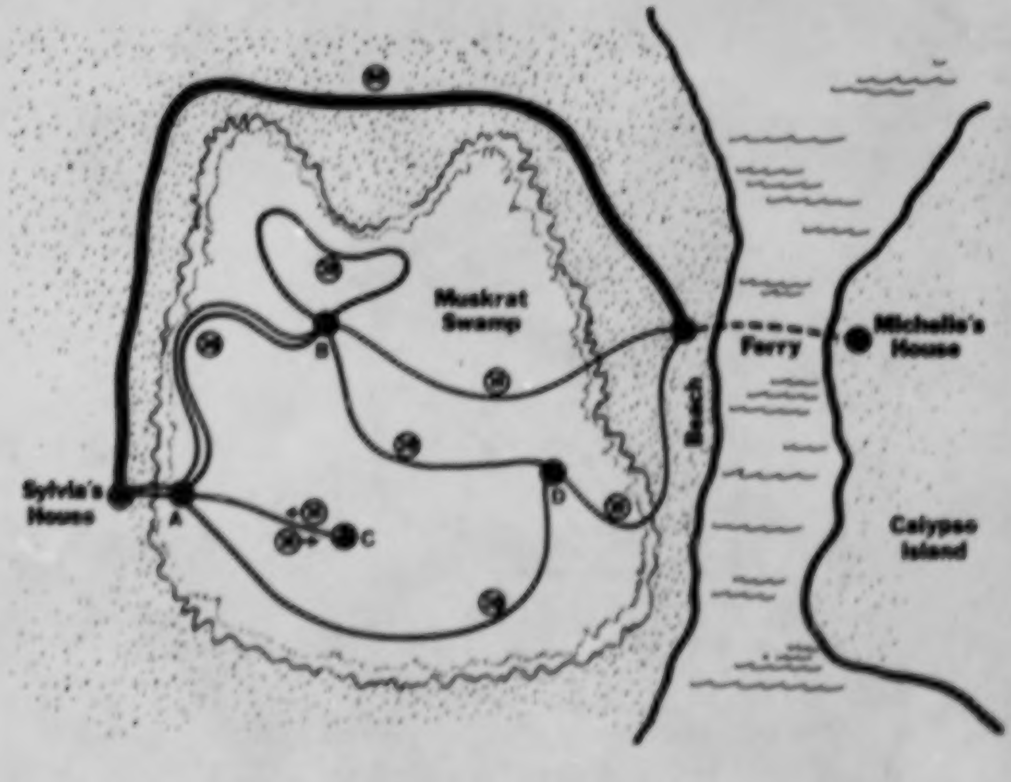
Invite a student to divide the square into thirds and suggest that the student use a meter stick for accuracy. For example, if the square is 71 cm wide, each section should be about 24 cm wide.

Label the regions "AB_{30}", "ACA_{30}", and "AD_{30}" to indicate the path and the time it would take to ride along that path.


AB_{30}
ACA_{30}
AD_{30}

T (pointing to "AB_{30}"): In each region, I indicate a path Sylvia could choose and the number of minutes to ride that path. Suppose Sylvia chooses the path from A to B.

Draw the path from A to B in red on the poster.


Muskrat Swamp
B
Sylvia's House
A
C
D
Ferry
Beach
Michelle's House
Calypso Island

P 6

T: Now Sylvia is at B. How many paths could she take from B?

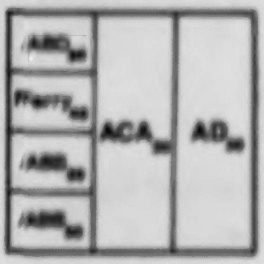
S: Four; she could take the path from B to D or from B to the ferry. Also, there are two ways to go around the loop back to B. She won't go back to A.

T: How do I show this on the square?

S: Divide the region labeled "AB_{30}" into fourths.

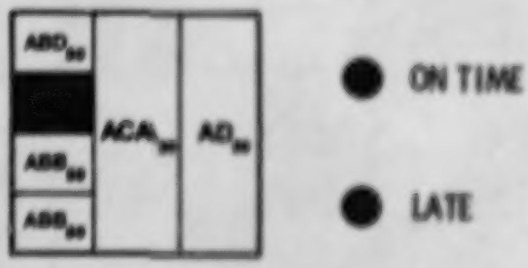
Ask the students to help you divide the region and then to label its four parts.

NOTE: The labels should indicate the entire path taken thus far and the total time.



ABD_{50}
$Ferry_{40}$
ABB_{60}
ABB_{60}
ACA_{30}
AD_{30}

Color the region labeled "$Ferry_{40}$" blue and write the following code near the square.



ABD_{50}
ACA_{30}
AD_{30}
ABB_{60}
ABB_{60}
ON TIME
LATE

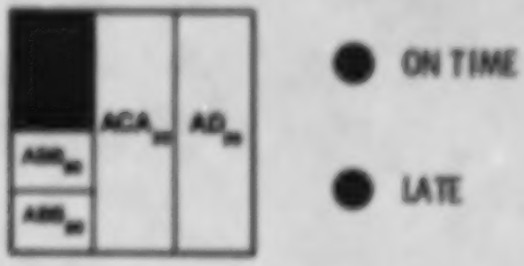
Point to the region labeled "ABD_{50}". Trace a path from A to B to D.

T: If Sylvia rides from A to B to D, she uses up 50 minutes. What are her chances of getting to the ferry on time?

S: One out of two. She could take the path to the ferry and arrive just in time (60 minutes), or she could ride to A and use up too much time (70 minutes).

T: How could I show that on the square?

S: Divide the region labeled "ABD_m" in half; color one-half red and one-half blue.



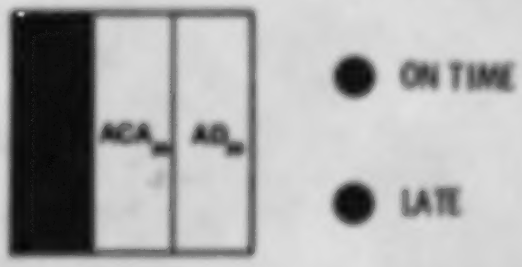
Point to the regions labeled "ABB_m" and trace the two paths from A to B and around the loop (once in each direction) to B.

T: If Sylvia takes one of these paths and uses up 50 minutes, what are her chances of getting to the ferry on time?

S: 1 out of 2. She could either go from B directly to the ferry and arrive on time (60 minutes) or go from B to D and use up too much time (70 minutes). She wouldn't go back to A and wouldn't go around the loop again.

T: How do I show this on the square?

S: Divide each region labeled "ABB_m" into two pieces of the same size; color one piece red and one piece blue.



P 6

Refer to the third of the square that is colored.

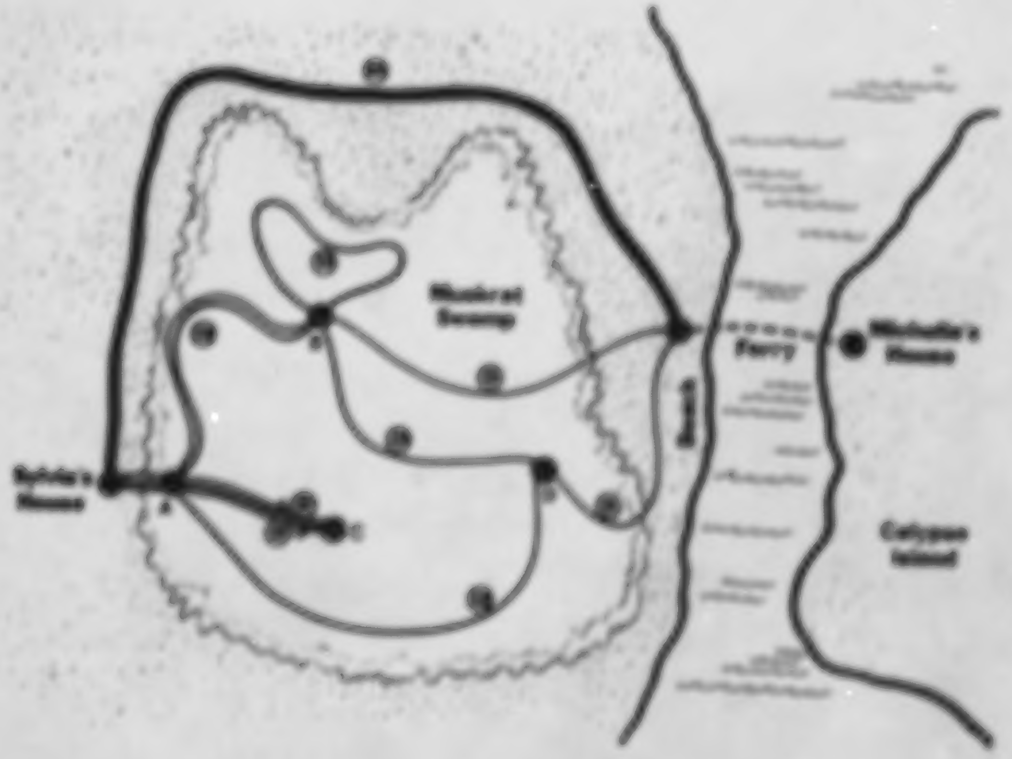
T: If Sylvia first chooses the path from A to B, is she more likely to be on time or late for the ferry?

S: On time; more than half the region is colored blue.

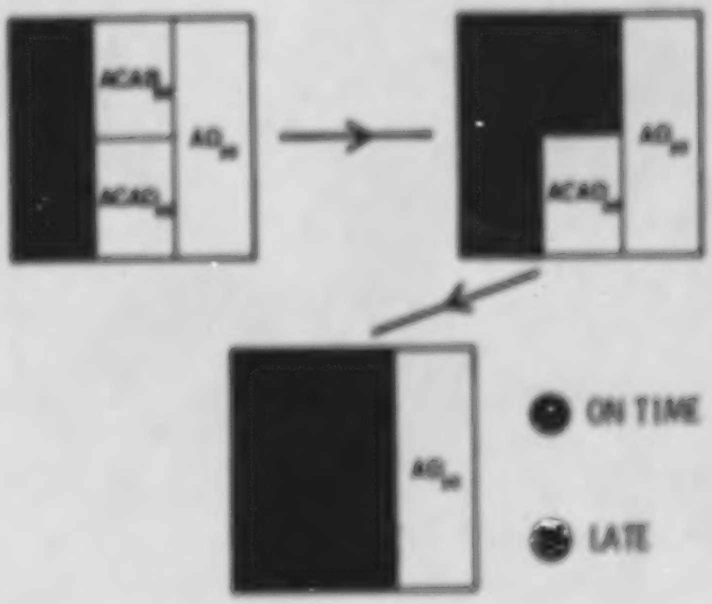
S: On time; there are five small blue pieces and three small red pieces.

T (pointing to the region labeled "ACA_m"): Let's calculate Sylvia's chances of being on time if she first chooses the dead-end path to C and then comes back to A.

Draw this path in blue on the poster.


Sylvia's House
A
B
C
D
Muskrat Swamp
Ferry
Michelle's House
Calypso Island

Analyse this possibility in a similar manner as for "AB_m". The following sequence of squares indicates the steps your students might suggest.

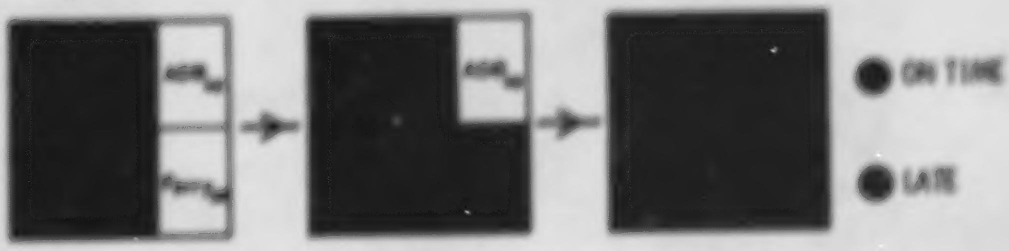


T (pointing to the region labeled "AD_m"): We still have to calculate Sylvia's chances of being on time if she first chooses the path from A to D.

Draw a path from A to D in green on the poster.

T: On your own paper, try to decide how we should color the last third of the square.

Let the students work about five minutes, helping those who are having difficulty. Then complete the problem collectively at the board. The following sequence of squares indicates the steps your students might suggest.



P 6

T: We analyzed all the different paths Sylvia could take in 60 minutes or less and determined for each if she would get to the ferry on time. Is she more likely to be late or on time?

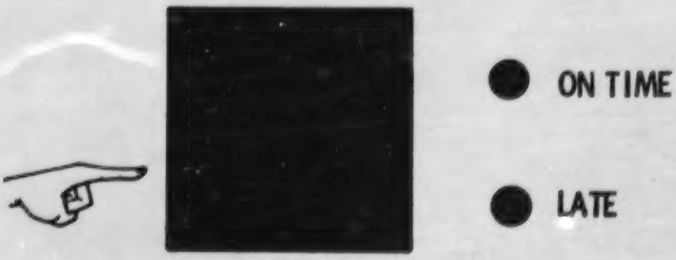
S: The chances are about the same because about half the square is red and about half is blue.

T: Can we calculate exactly what Sylvia's chances are to be on time?

Let the students discuss this question. Encourage any suggestion to compare the number of small pieces which fit into the entire red region to the number of small pieces which fit into the entire blue region. If necessary, ask the following questions.

T: Let's compare the amount of the square colored red to the amount colored blue. Where is one of the smallest pieces in our division of the square?

Invite a student to point to one of the smallest pieces; for example,



T: How many pieces of this size fit into the red region?

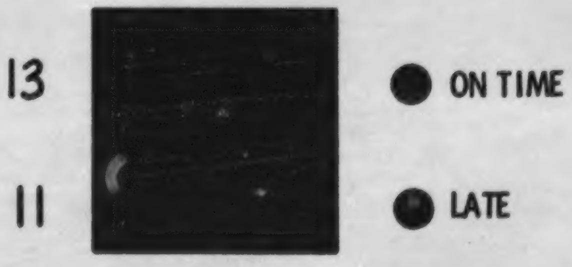
S: Eleven.

Ask the student to show how this was calculated. If it would be helpful, subdivide the red region into pieces all the same size. Write "11" in red near the square.

T: How many pieces of this size fit into the blue region?

S: Thirteen.

Write "13" in blue near the square.

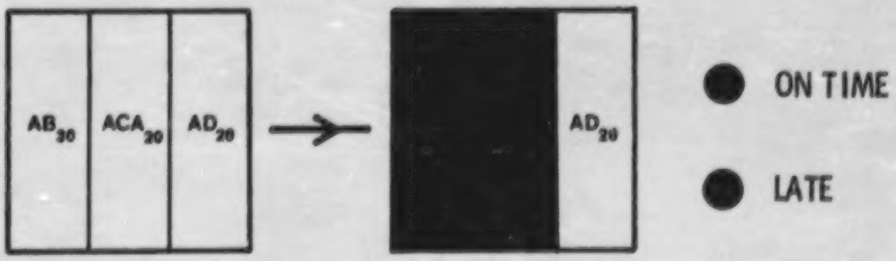


T: What is Sylvia's probability of getting to the ferry on time?

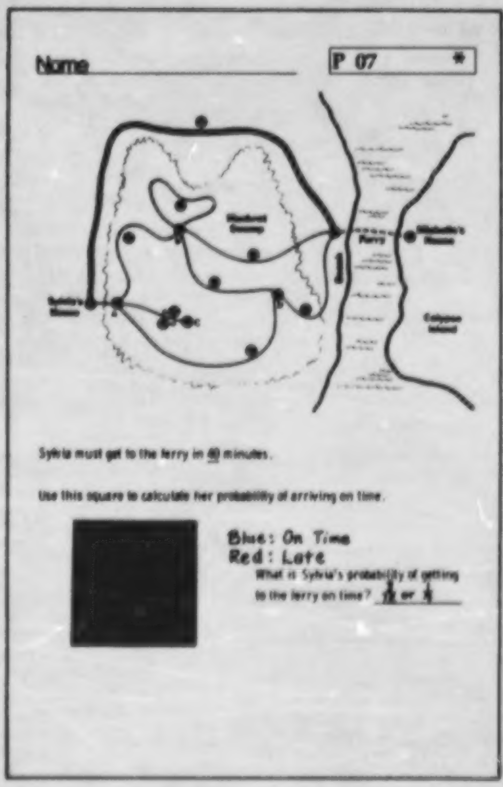
S: $\frac{13}{24}$ or 13 out of 24.

Compare the solution, $\frac{13}{24}$, with the students' estimates of Sylvia's chances.

Worksheets P07 and P08 are available. Point out to the students that the map on Worksheet P07 is the same as the one used in class and that the problems are similar except that in this one Sylvia has only 40 minutes to get to the ferry. After the students have worked about five minutes, stop the individual work and solve part of the problem on Worksheet P07 collectively to be sure that everyone knows how to start. The following sequence of squares indicates some steps your students might suggest.



At this point, instruct the students to complete this problem on their own.



Name

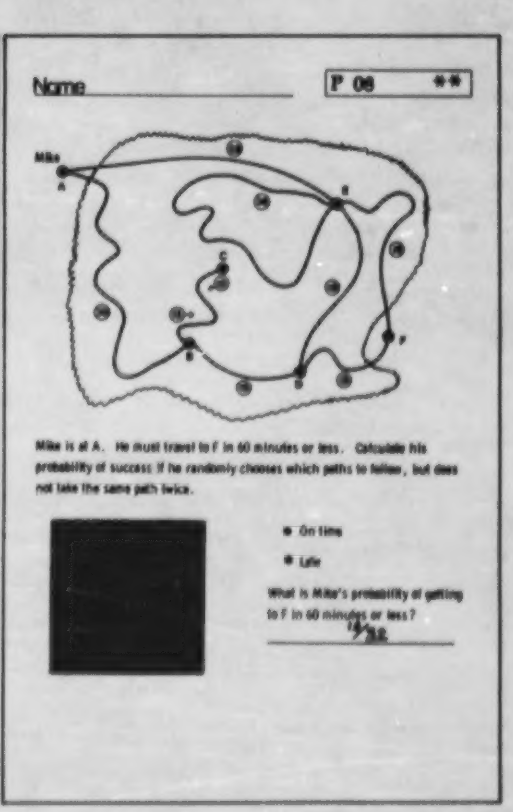
P 07 *

Sylvia must get to the ferry in 60 minutes.

Use this square to calculate her probability of arriving on time.

Blue: On Time
Red: Late

What is Sylvia's probability of getting to the ferry on time? [illegible] or [illegible]


Mike
A
E
C
D
B
G
F

Name

P 08 **

Mike is at A. He must travel to F in 60 minutes or less. Calculate his probability of success if he randomly chooses which paths to follow, but does not take the same path twice.

● On time

● Late

What is Mike's probability of getting to F in 60 minutes or less? 14/32

2.4 A Fortune Teller's Device

CAPSULE LESSON SUMMARY

Calculate the probability of forming one long piece of rope when some of the ends of three ropes are joined by two knots. Vary how the ropes are held and analyze the new situation which results.

MATERIALS

Teacher :	Three identical pieces of light rope or cord, each about one meter long; meterstick; colored chalk
Student :	Ruler; colored pencils; paper; Worksheets P09 and P10

DESCRIPTION OF LESSON

Exercise 1

Initiate a discussion about fortune tellers.

T : Do you know of any fortune tellers? What kinds of questions do people ask fortune tellers? What do fortune tellers use to predict the future?

Once you have gained the students' interest, tell the following story. Have the three pieces of rope on hand.

T : My story is about a fortune teller who usually uses a crystal ball and cards to predict the future of her clients. Many lovers come to her to ask about their future when they get married. For these romantic youths, the fortune teller has a special way to foretell the future of proposed marriages. She uses three identical pieces of rope like the ones I have here.

Show the class the three pieces of rope. Then fold the three ropes in half and hold them as shown in the next illustration. Twist the strands slightly in your hand so that the students cannot tell which ends belong to the same piece of rope.

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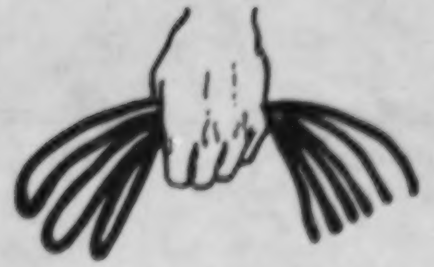
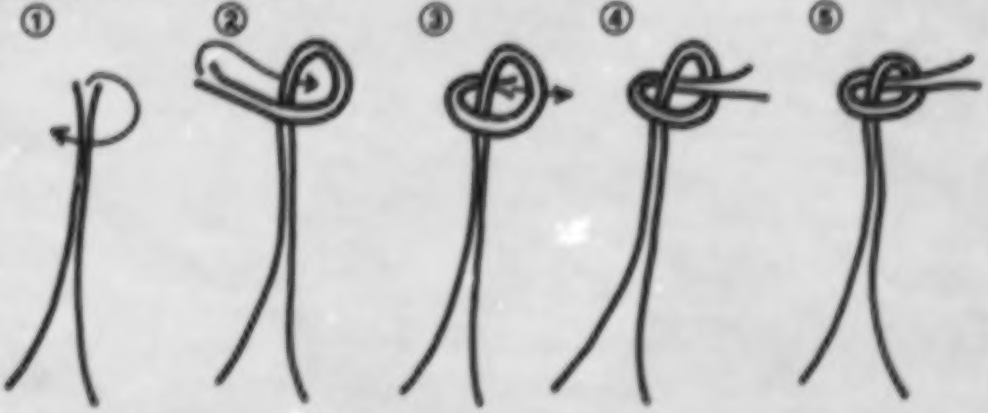


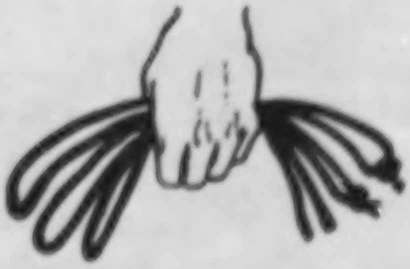
T: When young lovers ask this fortune teller whether or not a proposed marriage will be long and happy, the fortune teller holds the three ropes like I am holding them now. She tells the couple to select two of the six loose ends and to tie them together. Then she has them tie a second knot using two of the remaining loose ends. Let's try it.

Select students to play the role of the couple. Suggest that a simple overhand knot be used (see the illustration below).

Place the two ends to be tied parallel to each other.



After two knots are tied, there should still be two loose ends.



T: After two knots are tied, the fortune teller releases the ropes. What could happen?

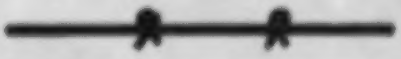
S: There might be one long piece of rope.

S: There might be some loops.

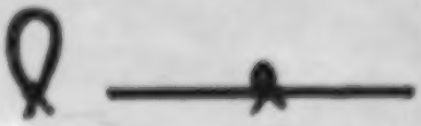
T: The young lovers' future marriage will be long and happy if the result is one long piece of rope. Let's check our ropes.

Release and untangle the ropes. Determine which of the following outcomes occurs.

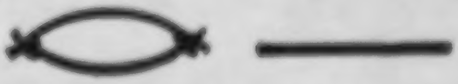
- one long piece



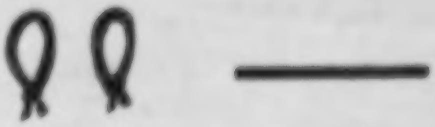
- one small loop and one double piece



- one large loop and one single piece



- two small loops and one single piece



Note whether or not the fortune teller predicts a long and happy marriage; that is, whether or not one long piece of rope occurs. Untie the ropes and repeat the experiment two or three more times.

T: What do you think is the probability or chances that the rope will be one long piece after two knots are tied?

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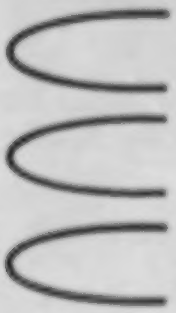
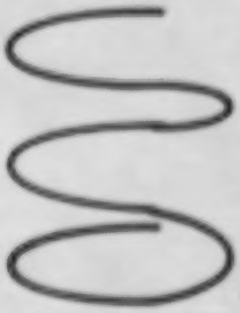
List the predictions of several students. Point out that only a few trials of the experiment have been made in class and that those results do not necessarily indicate the probability of success for one long piece.

Draw the following picture on the board.



T: This is a picture of the three ropes. To analyze what happens when two knots are tied, we'll pretend that we can distinguish the three ropes. I've drawn the ropes in different colors so that it is easier to talk about them.

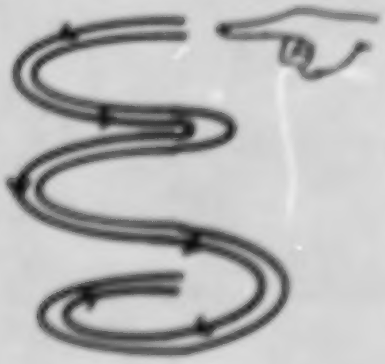
Add the following information to the picture.



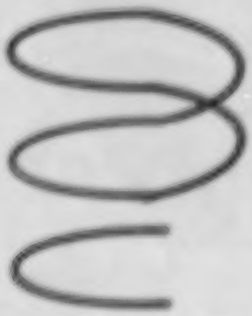
T: The connections show which ends are joined by the two knots. Would these two knots make one long piece of rope?

S: Yes.

Invite a student to trace the long piece of rope by starting at one loose end and following the ropes and connections until the other loose end is reached; for example,



Change the connections in the picture.

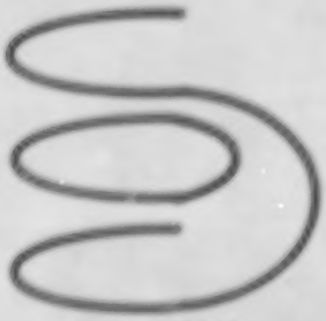


T: If these two knots were tied, would one long piece of rope result?

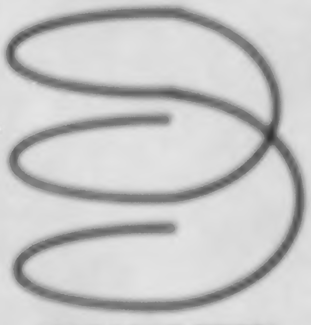
S: No, the green rope is not attached.

S: No, the red and blue ropes form one big loop.

Discuss the following two situations in a similar manner.



(TWO PIECES)



(ONE LONG PIECE)

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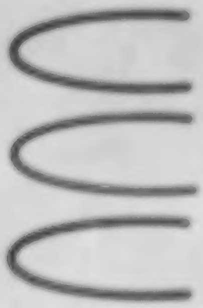
Erase the connections and draw a square.

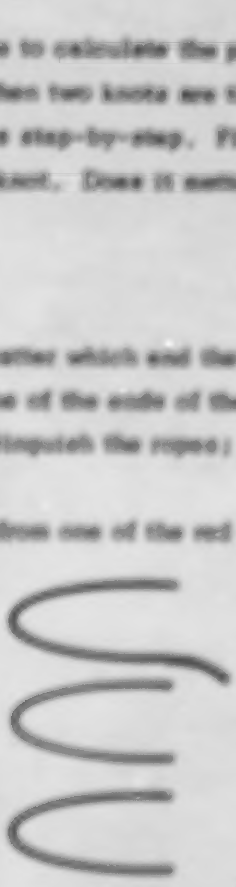


T: We'll use a square to calculate the probability that one long piece of rope will result when two knots are tied randomly. Let's analyze the knot-tiers' actions step-by-step. First, the couple selects one end to begin the first knot. Does it matter which end they start with?

S: No.

T: Since it doesn't matter which end they start with, let's say that the couple chooses one of the ends of the red rope. Remember, the couple cannot distinguish the ropes; they are not colored.

Begin drawing a connection from one of the red ends.



T: Next they tie this end to one of the other loose ends. In how many different ways can the first knot be completed?

S: Five, since there are five other loose ends: one red end, two blue ends, and two green ends.

T: Would any of these five possible first knots be lucky or unlucky for the knot-tiers?

S: Yes, tying the two red ends together would be unlucky since a loop would be formed.

T: Tying the red end to a blue end or to a green end would be alright for the first knot since no loop is formed. But we don't know what will happen when a second knot is made.

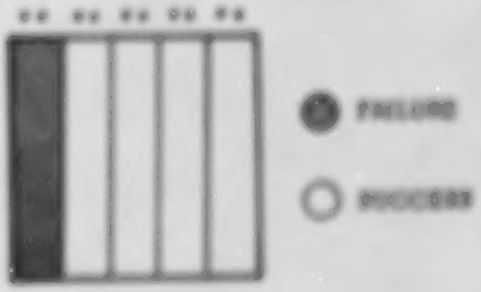
T: What is the probability that the knot-tiers would be unlucky and would form a loop with the first knot?

S: $\frac{1}{5}$ or 1 out of 5.

T: How can I show this on the square?

S: Divide the square into five parts of the same size. Color one region to indicate that a loop is formed and that one long piece of rope cannot result.

Invite a student to divide the square into five parts of the same size using a meter stick for accuracy. Indicate a code (shaded or unshaded) next to the picture.



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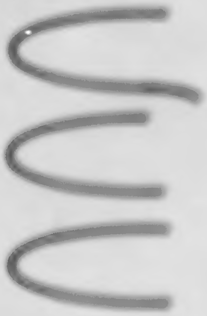
T: I shaded the first region to show that failure results if the first knot joins the two red ends. Now we must study what happens if the first knot joins a red end to one of the other four ends.

Refer to the picture of the three ropes.



T: Which is luckier for the knot-tier: to tie the red end to a blue end for the first knot or to tie the red end to a green end? Or does it matter?

S: It doesn't matter. There's no difference between the blue rope and the green rope.

T: If the red end is tied to a blue end, does it matter *which* blue end it is tied to?

S: No.

T: If the red end is tied to a green end, does it matter which *green* end it is tied to?

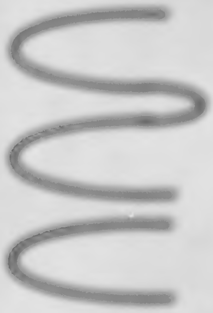
S: No.

T (pointing to the four unshaded regions): So the probability of success will be the same for each of these four knots. We need only to determine how to shade one of these regions; the other three regions will be the same.

Point to the leftmost unshaded region labeled "● ●".

T: Let's determine how to shade this region; that is, let's calculate the probability of success if the red end is joined to one of the blue ends.

Represent this situation by completing the connection from the red end to a blue end.



T: How many different second knots could be tied?

Invite students to the board to show all the possibilities for the second knot. Continue until the correct number, six, is agreed upon and a student has shown the six different knots in a systematic way.

Distribute copies of Worksheet P9.

T: The six different ways the second knot could be tied are shown on this worksheet. For each picture, determine whether or not the two knots make one long piece of rope.

Continue the collective lesson after most students have completed the worksheet.

T: Look at your worksheet. In how many of the six possibilities is one long piece of rope formed?

S: Four.

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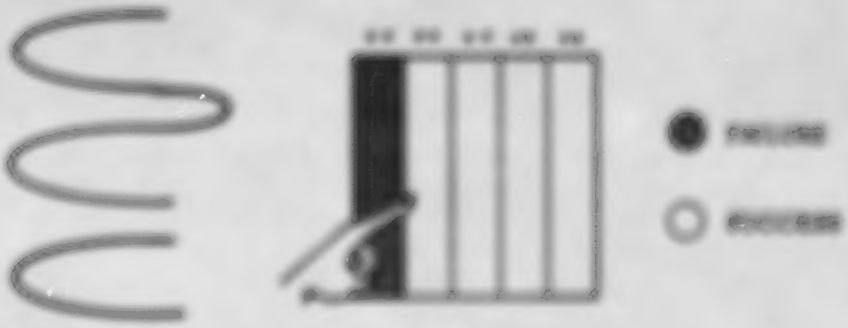


T (pointing to the leftmost unshaded region): We are now trying to shade this region. The first knot joins a red end to a blue end. What is the probability that the second knot will produce one piece of rope?

S: $\frac{4}{6}$ or 4 out of 6. We could also say "$\frac{2}{3}$" since $\frac{4}{6} = \frac{2}{3}$.

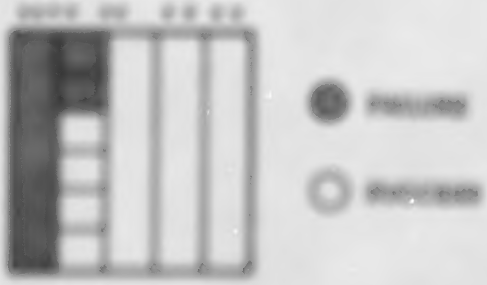
T: Explain your answer.

S: We found that there were six different ways to tie the second knot and that four of those six ways resulted in one piece of rope.

T: How should I shade the region?

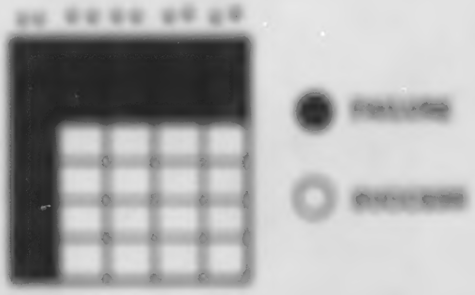
S: Divide the region into six parts of the same size and shade two parts for failure and leave four parts unshaded for success.

S: It's the same if you divide that region into three parts and shade one part and leave two parts unshaded.



T: How should I shade the other three regions?

S: The same as the region we just shaded since we agreed earlier that the probability of success would be the same for each of these four regions.



T: The whole square is completed, so we can now calculate the probability that one piece of rope will result after two knots are tied. In the square, how many chances of success are there? How many chances of failure . . . ?

S: Sixteen chances of success and fourteen of failure.

T: What is the probability of success?

S: $\frac{16}{30}$ or 16 out of 30, since 16 + 14 = 30. We could also say "$\frac{8}{15}$" because $\frac{16}{30} = \frac{8}{15}$.

T: Is success or failure more likely?

S: Success, since 16 is larger than 14.

S: Success, since $\frac{16}{30}$ is a little larger than $\frac{1}{2}$.

Compare the students' predictions to the result, $\frac{16}{30}$.

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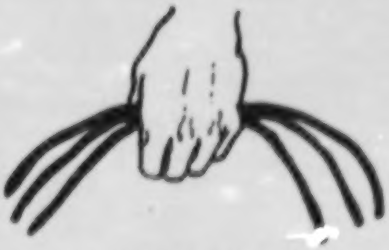
Exercise 2

T: The young lovers were very disappointed if the fortune teller predicted that their future marriage would not be long and happy. Many of these people would ask the fortune teller to let them try again.

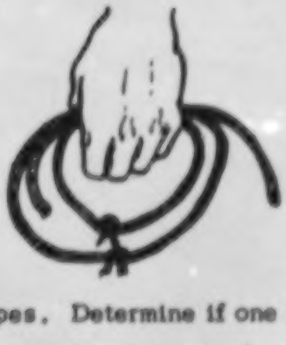
The fortune teller would agree to give them another chance, but she would use a different method. She would hold the three ropes like this.

Hold the three ropes as in the following picture. Twist the ropes slightly in your hand so that the students cannot tell if two ends belong to the same rope.



T: The goal is the same: to form one long piece of rope by tying two knots. But this time the ropes are held differently and each knot must join a loose end from the right side to a loose end from the left side. Who would like to try this new method?

Select a student to tie two knots according to your instructions; for example,



Release and untangle the ropes. Determine if one piece of rope is formed. Untie the ropes and repeat the experiment two or three more times.

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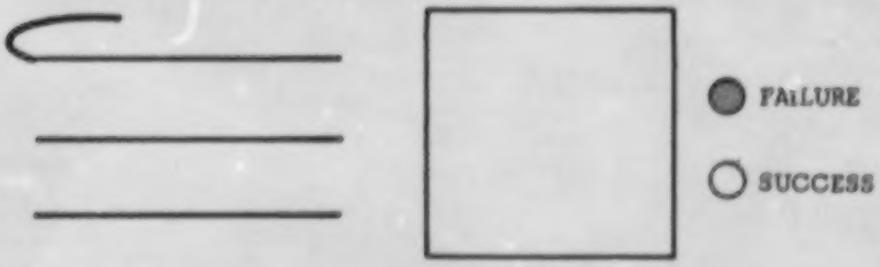
T: What do you think the probability of success is this time? Which of the two methods do you think has the higher probability of success?

List the students' predictions on the board. Distribute Worksheet P10 and draw the following picture on the board.



T: I'd like you to calculate the probability of success using the fortune teller's second method. I've drawn the three ropes straight this time to represent how the ropes are held. To tie the first knot, does it matter which end we start with?

S: No.

T: Let's everyone start with the red end on the left side. Remember that the new method requires that each knot must join one end from the left side to one end from the right side. Now on the worksheet calculate the probability of success.

Let the students work independently for a few minutes helping those who need assistance. Then do the first steps of the problem collectively.

T: What should we do first?

S: Divide the square into thirds since the red end on the left can be joined to one of the three ends on the right.

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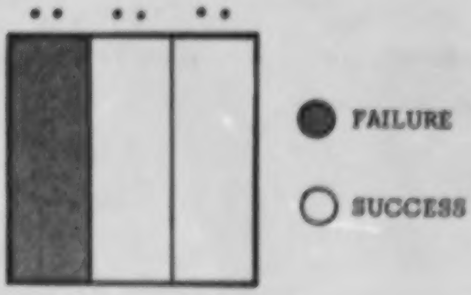
S : The region for red joined to red can be shaded since the first knot forms a loop which means failure.



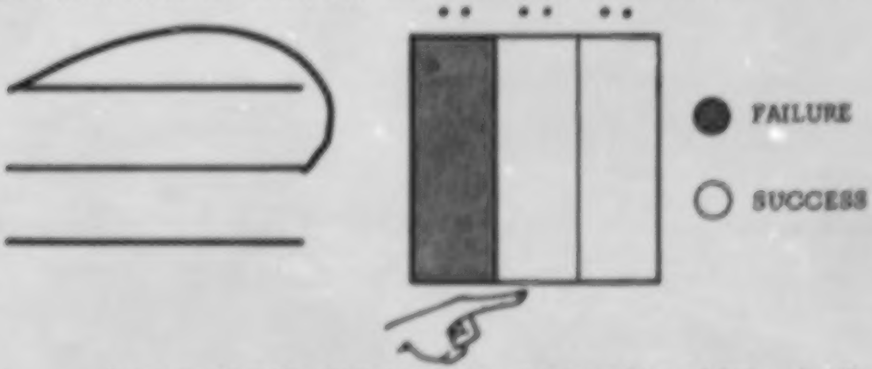
T : Complete the problem on your own by determining what happens when the first knot joins a red end to either a blue end or a green end.

After many students are finished, complete the solution collectively.

T : Does it matter whether the red end is joined to the blue end or the green end?

S : No, the ropes are identical. The probability of success will be the same for both cases.

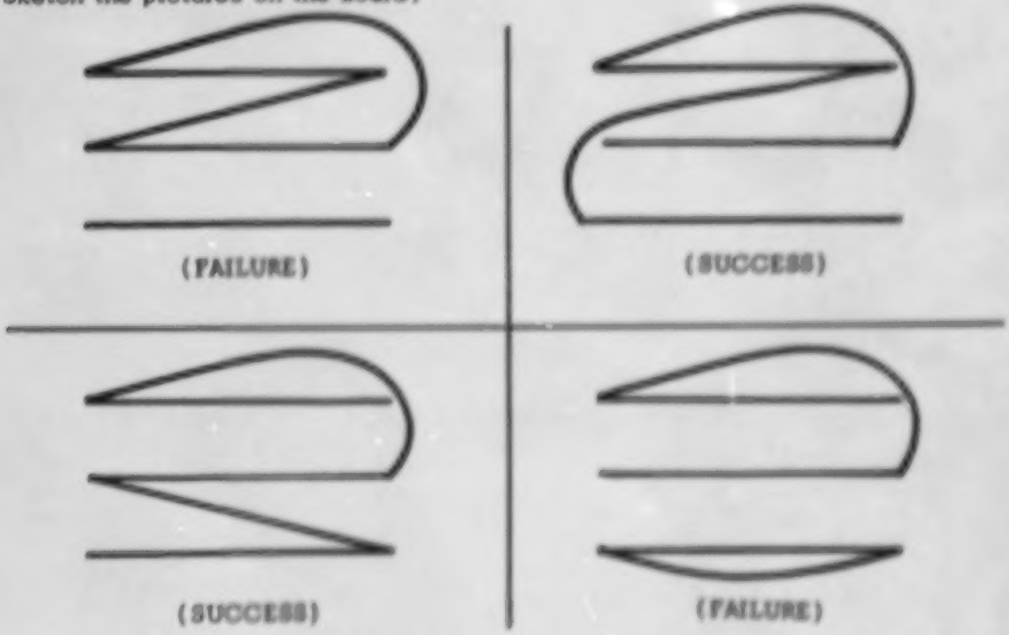
Add a connection to the picture and point to the region headed "● ●".



T : Suppose that the first knot joins a red end to a blue end. How many different second knots could be tied?

S : Four.

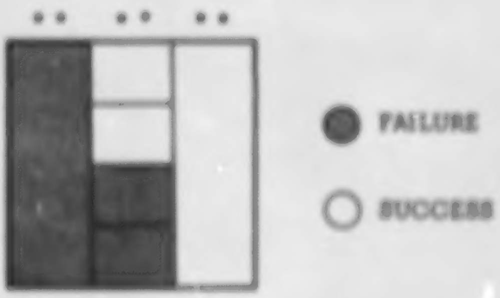
Invite a student to trace the four possible second knots as shown below. Sketch the pictures on the board.



T: Of these four possibilities, which result in success and which result in failure?

The class should determine that two possibilities result in success and two result in failure as indicated in the preceding illustration.

Invite a student to divide and shade the middle region of the square, corresponding to the class' findings.



T: How should we shade this last region for "● ●"?

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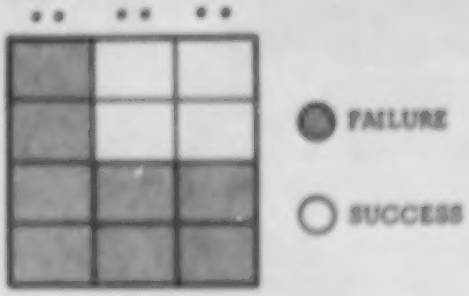
S: The same as the region for "● ●" since we agreed earlier that it doesn't matter whether the first knot joins the red end to a blue end or to a green end.



T: What are the chances for success? . . . failure?

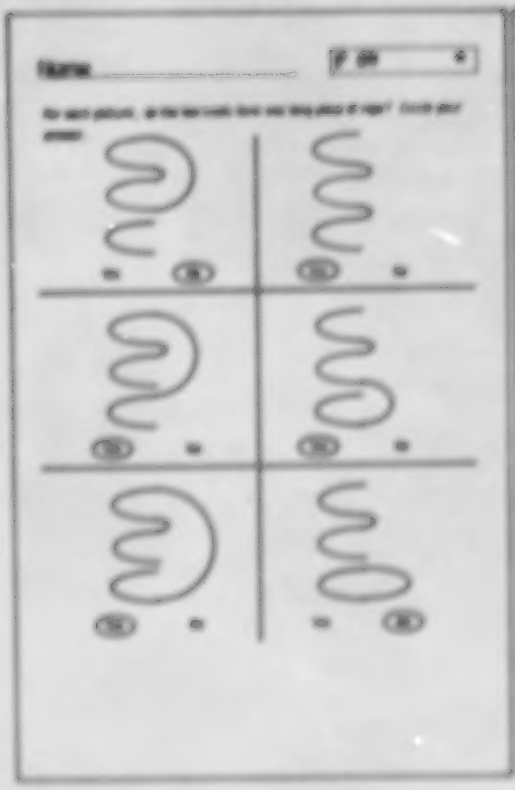
S: Eight chances for failure and four chances for success.

T: What is the probability of success?

S: $\frac{4}{12}$ or 4 out of 12, or $\frac{1}{3}$ since $\frac{4}{12} = \frac{1}{3}$.

T: Which method of the fortune teller's is better for the young lovers?

S: The first method since $\frac{16}{30}$, which is about $\frac{1}{2}$, is larger than $\frac{1}{3}$.



Name

Name

3

PROBABILITY GAMES

The lessons of this chapter deal with a game for two people played with marbles. Three marbles, two red and one blue, are used and from them one marble is chosen at random. If the red marble is chosen, player A scores a point and if the blue marble is chosen, player B scores a point. The player to accumulate ten points first is the winner. Analysis quickly reveals that this game is not fair so the rules change: If the red marble is chosen, player A scores one point but if the blue marble is chosen, player B scores two points.

A geometric, measure-minded method is used to analyze situations where an intermediate score is known and both players' chances for winning is to be computed. The problems are essentially the same as those in which one must decide how to share the stake of a probabilistic situation according to the probabilities involved. This new method is a generalization of the combinatorial or "tree" method. Furthermore, it provides another opportunity for students to explore patterns in sequences of numbers.

CAPSULE LESSON SUMMARIES

Section 3.1:	Introduce a probability game played by two people. Calculate the winning chances for the two players starting from intermediate scores. Record these chances on a chart.
Section 3.2:	Review the probability game introduced in Section 3.1 and recall how to read the chart on Poster #3. Calculate the winning chances for the two players starting from intermediate scores and complete the chart. In some cases predict the results using patterns observed in the numbers already recorded in the chart. Then check if your predictions are correct. (Worksheet P11)

P 4

1.1 Probability Games #1

CAPSULE LESSON SUMMARY

Introduce a probability game played by two people. Calculate the winning chances for the two players starting from intermediate scores. Record these chances on a chart.

MATERIALS

Teacher:	Probability Poster #1 for IG-V; Scorecards for Probability Games; meter stick; colored chalk; one blue and two red marbles; small paper bag
Student:	Paper; colored pencils; ruler

NOTE: If you have a magnetic chalkboard available, prepare the scorecards by sticking a small piece of magnetic material (provided in the classroom set of materials) on the back of each card. Otherwise, prepare scorecards to stick to the chalkboard by attaching a loop of masking tape or a small wad of plastic caulking compound (e.g. Mortite or Rope Caulk) to the back of each card. These scorecards are needed in Exercise 3.

DESCRIPTION OF LESSON

Display Probability Poster #1 for IG-V on one side of the board.

Exercise 1

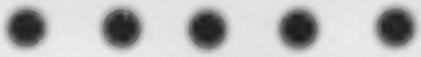
Write this information on the board.

● Rita ● Bruce

Use two red marbles and one blue marble to demonstrate the game being played by Rita and Bruce as you describe it. Place the marbles in a small paper bag.

T: Rita and Bruce are playing a game. Bruce has two red marbles and one blue marble in a small bag. Without looking, Rita selects a marble from the bag. If she chooses a red marble, she scores a point; if she chooses the blue marble, Bruce scores a point. Remember, red for Rita and blue for Bruce.

Demonstrate the game by shaking the three marbles in the small bag and by asking a student to select one marble without looking. Draw one red dot or one blue dot on the board according to the color of the marble chosen. Repeat the process about four more times. For example, if the marbles appeared in this order: red, blue, red, red, blue, you would draw the following dots.



T: What is the score now?

S: Three points for Rita and two points for Bruce.

The answer depends, of course, on the color of marbles selected in your class.

T: Do you think this is a fair game?

S: No, there are two red marbles and only one blue marble. Rita should score a point more often than Bruce.

T: How could Rita and Bruce make the game fair if they still wish to use two red marbles and one blue marble?

Elicit several suggestions from the class. If no one suggests the following scoring method, do so yourself.

T: Rita and Bruce agree to the following rules. The game starts at a score of 0-0 (read this score as "0 to 0"). Each time a red marble is selected, Rita scores one point. Each time the blue marble is selected, Bruce scores two points. The winner is the first person to score ten points.

Point to the row of colored dots indicating the results of trials your class made.

T: Using this scoring method, what would be the score of the game we have already begun?

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The ensuing discussion will depend on the results obtained in your classroom. It might be as follows.

S: Three for Rita and four for Bruce. Rita receives one point each time a red marble is selected and that happened three times. Bruce receives two points each time the blue marble is selected, and that happened twice; so Bruce would have four points.

Record the score on the board, for example,

● Rita ● Bruce
3 – 4

T: Yes, the score is 3-4. We will always give Rita's score first.

Shake the three marbles in the bag and again invite a student to select a marble. Add an appropriate color dot to your row.

● ● ● ● ● ●

T: A blue marble was selected. What is the score now?

S: 3-6, because Bruce receives two points for the blue marble being selected.

Insist that the student announces Rita's score first; i.e., the score is now 3-6, not 6-3. Replace "3-4" with "3-6" on the board and draw this card picture.

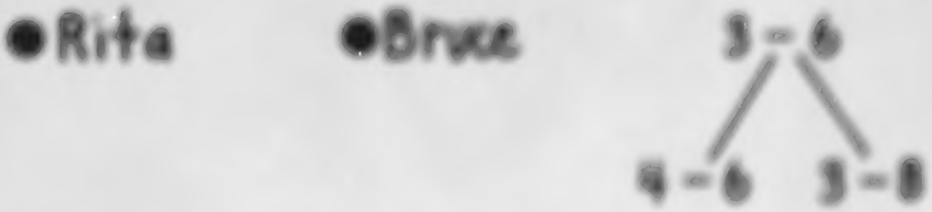
● Rita ● Bruce
3 – 6

T: The score is now 3-6. What could the next score be?

S: 4-6, if a red marble is selected.

S: 3-8, if a blue marble is selected.

Add this information to the next picture on the board.

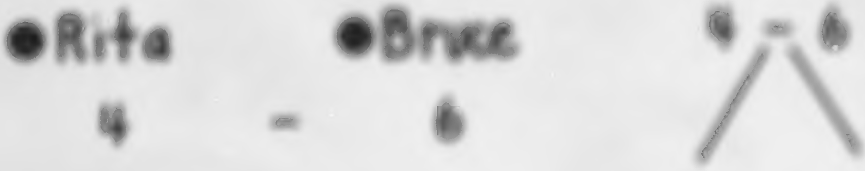

Rita
Bruce
3 - 6
4 - 6
3 - 8

Continue playing the game with your class. A red marble might be chosen.

T: What is the score now?

S: 4-6.

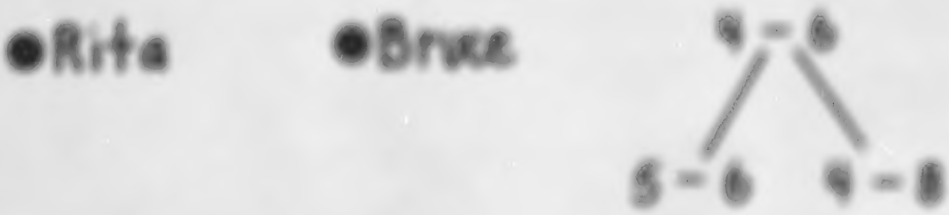
Record the score and change the next picture.


Rita
Bruce
4
-
6
4 - 6

T: What could the next score be?

S: 5-6 or 4-8.

Record this information in the next picture.


Rita
Bruce
4 - 6
5 - 6
4 - 8

P 4

Continue this activity until your class understands the rules and the scoring system of the game.

T: Do you think this is a fair game?

S: It's fair since there are two red marbles for Rita and since Bruce scores two points each time the blue marble is selected.

S: We don't know if it is exactly fair.

T: Let's study this game some more.

Exercise 2

NOTE: In Exercise 2 your class fills in the places in Probability Poster #1. The corresponding intersection arrow is indicated before the description of an activity.

GAME IS AT 8-8

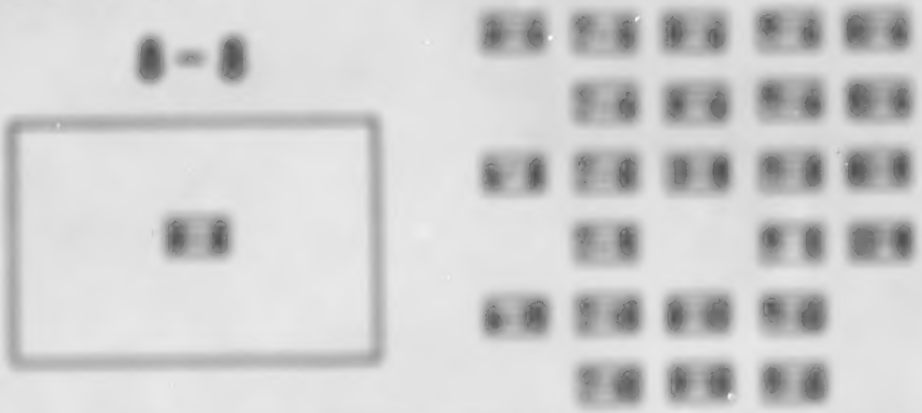
T: One afternoon Rita and Bruce played this game until they had to stop. But their game was not over; they stopped when the score was 8-8 and agreed to continue the next day. When they continue the next day, who is favored to win? Remember, they play until one of them scores ten points.

S: Bruce, because he wins if the blue marble is selected one more time whereas Rita wins if a red marble is selected two more times.

S: It could be Rita because there are two red marbles but only one blue marble.

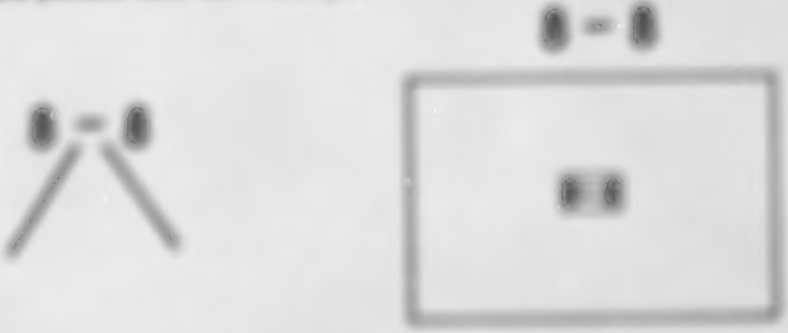
S: Maybe Rita and Bruce have equal chances to win.

Draw a large rectangle on the board. Place the scorecard 8-8 in the rectangle and display the other scorecards to one side of the board. Write "8-8" above the rectangle.



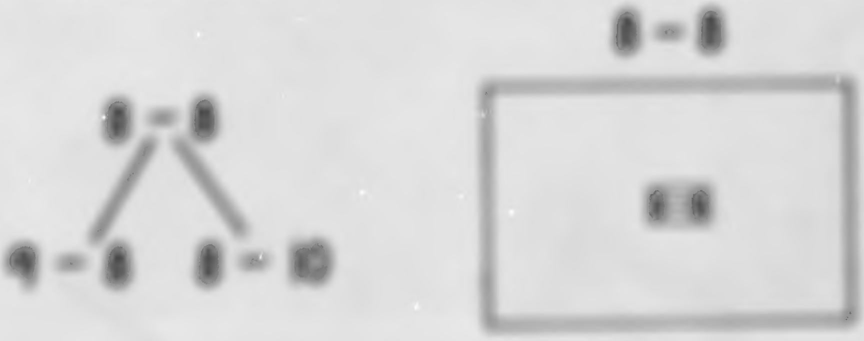
T: We will use this rectangle to calculate [illegible] and [illegible] chances of winning when the score is 8-8.

Draw this card picture near the rectangle.



T: If the score is 8-8, what could the next score be?

S: 9-8 if a red marble is selected or 8-10 if the blue marble is selected.



P 4

[illegible]

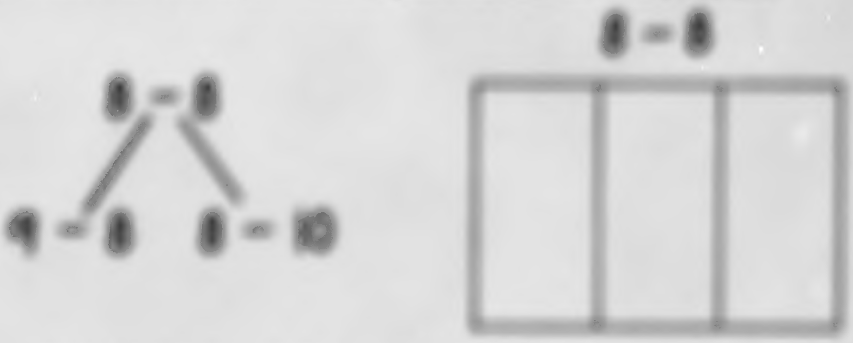
[illegible]

[illegible]

[illegible]

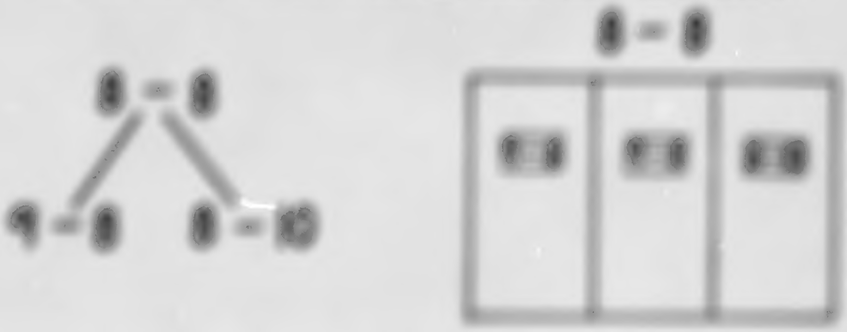
[illegible]

[illegible]



[illegible]

[illegible]

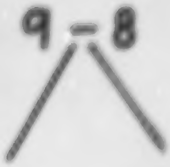

8 – 8
9 – 8
8 – 10
8 – 8
9 – 8
9 – 8

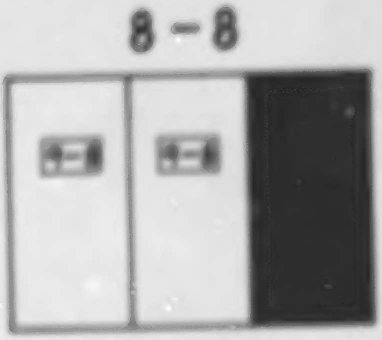
T: Now we must consider these three situations. What happens if the score is 8-10?

S: Bruce wins.

T: So we color the region for 8-10 blue to show that Bruce wins.

Color the region and adjust the card picture as shown below.

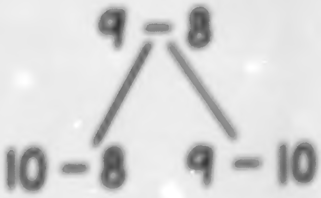


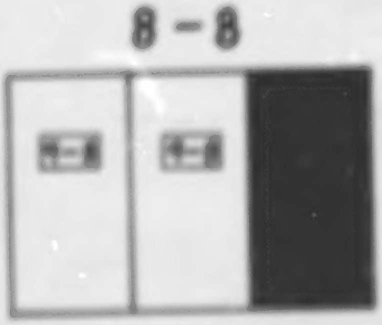


T: Now we must consider the possibilities when the score is 9-8. What could the next score be?

S: 10-8 and a win for Rita or 9-10 and a win for Bruce.

Add this information to the card picture.





T: What are the chances that the next score will be 10-8?

S: 2 out of 3, because there are two red marbles.

T: What are the chances that the next score will be 9-10?

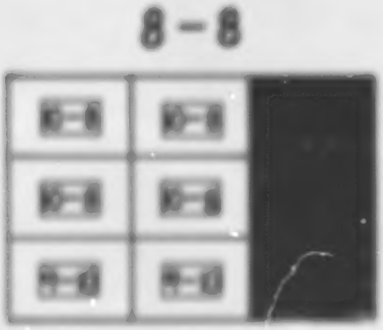
P 4

S: 1 out of 3, since there is one blue marble.

T (pointing to the regions for 8-8): How can I show that in these regions?

S: Divide each region into three pieces of the same size. Put [10-8] in two pieces and put [8-10] in one piece.

Invite a student to do the partitioning. Encourage the use of a meter stick.



Invite students to color the regions to indicate the winning players — red for Pam and blue for Bruce.

T: Who has the better chance of winning?

S: Bruce since more of the large rectangle is colored blue than is colored red.

T: How can we calculate the chances of winning for Bruce and for Rita?

Invite the class to discuss this question. Encourage comparing the part of the large rectangle shaded red to the part shaded blue. Lead the class to a method of comparing these parts as follows. Point to one of the smallest pieces of the partition.

8 - 8

T: One method of calculating the chances for Bruce and for Rita is to count how many pieces of this size (see preceding illustration) fit into the entire blue region and how many fit into the entire red region. How many pieces of this size could fit into the blue region?

S: Five.

Invite a student to the board to explain this answer. It may be useful to subdivide the larger blue region into three pieces. Then write "5" in blue near the rectangle.

T: How many pieces of this same size fit into the entire red region?

S: Four.

P 4

Write "4" in red near the rectangle.

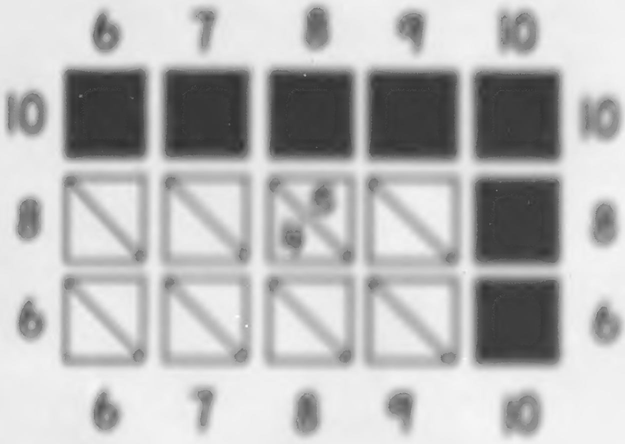
T: Therefore, if the score in a game is 8-8, Bruce has five chances to win and Rita has four chances to win. What is Bruce's probability of winning?

S: $\frac{5}{9}$ or 5 out of 9.

T: What is Rita's probability of winning?

S: $\frac{4}{9}$ or 4 out of 9.

Write "$\frac{5}{9}$" in blue and "$\frac{4}{9}$" in red near the rectangle. Record the result on the poster as shown below.



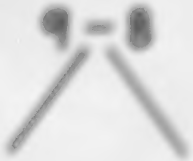
Indicate that you have put the result for the score 9-9 in the square which is in the column labeled "9" (for Rita's score) and the row labeled "9" (for Bruce's score).

Erase the board.

GAME IS AT 9-8

T: On another day the game was stopped when the score was 9-8.

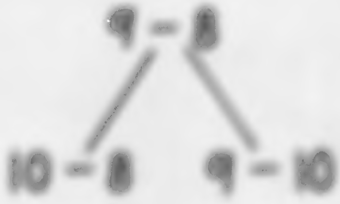
Draw this tree picture on the board.



T: Let's calculate Rita's and Bruce's winning chances when the game is continued the next day. What could the next score be?

S: 10-8 and a win for Rita if a red marble is chosen, or 9-10 and a win for Bruce if the blue marble is chosen.

Record this information.

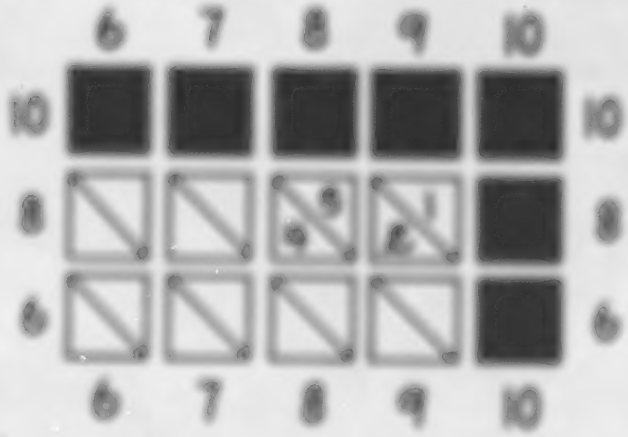


T: What are Rita's and Bruce's chances of winning?

S: Rita has two chances to win since there are two red marbles. Bruce has only one chance to win.

P 4

Invite a student to record this result for 9-8 on the poster. Explain that the result for 9-8 is placed in the column labeled "9" and the row labeled "8".

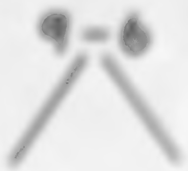


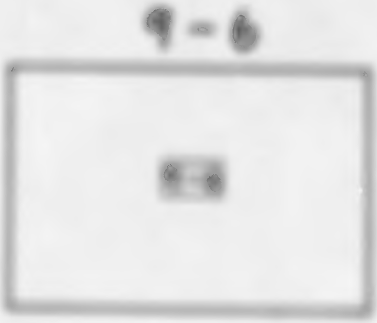
Erase the board.

GAME IS AT 9-6

T: On another day, the game was stopped when the score was 9-6.

Draw this road picture on the board. Also draw a rectangle and put the score-card 9-6 inside of it.





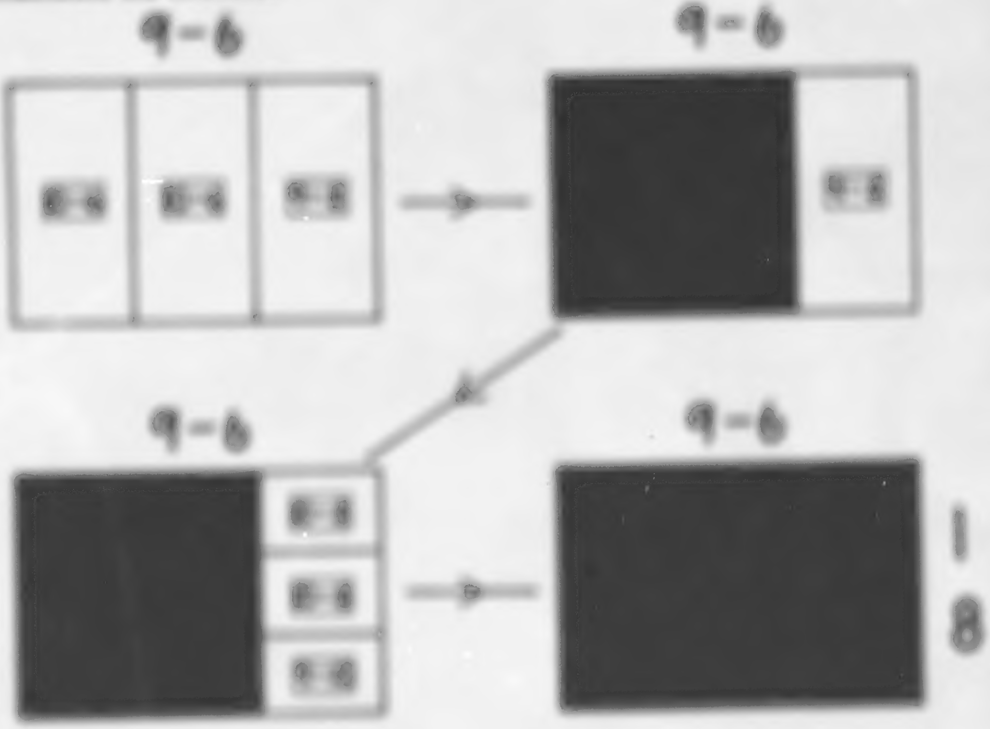
T: Who is favored to win this game? Why?

S: Nora, because she must score one more time (1 point) whereas Bruce must score two more times (4 points).

S: There are two red marbles and only one blue marble.

T: Draw a rectangle on your paper and calculate the winning chances for Rita and for Bruce when the score is 9-6. Use a ruler to divide the rectangle.

As you let the students work independently for a few minutes, assist any students who need help. Then use student suggestions to solve the problem collectively at the board. The sequence of rectangles below shows one way to calculate the chances.



T: What are Rita's and Bruce's winning chances when the score is 9-6?

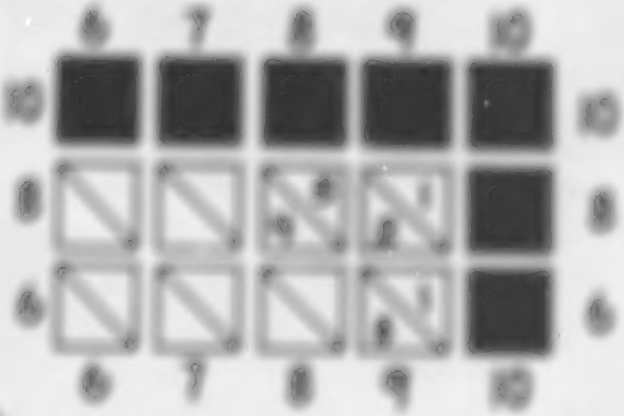
S: Eight chances for Rita and one chance for Bruce.

T: What is the probability that Rita will win?

S: $\frac{8}{9}$ or 8 out of 9.

P 4

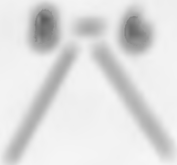
[illegible]


6
7
8
9
10
10
10
8
8
6
6
6
7
8
9
10

[illegible]

[illegible]

[illegible]



[illegible]

[illegible]

[illegible]

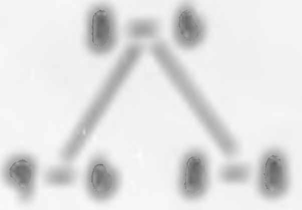
[illegible]

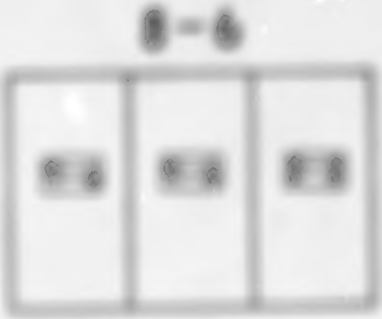
[illegible]

S: How do I show that on the rectangle?

S: Divide the rectangle into three pieces of the same size. Put [illegible] in two pieces and [illegible] in one piece.

Invite a student to do the partitioning.





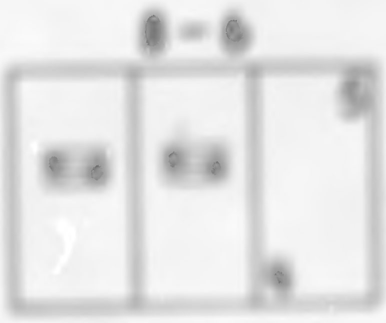
T (pointing to the region for [illegible]): What do we know if the score is [illegible]?

S: The next score could be [illegible] or [illegible]. You should divide the region for [illegible] into three sections of the same size.

T: Yes, we could calculate Bill's and Bruce's winning chances when the score is [illegible] that way. But do we have to do all that work?

S: No, the results for [illegible] are already recorded on the poster. If the score is [illegible], Bill has [illegible] chances to win and Bruce has [illegible] chances to win.

Invite a student to point to where this result is recorded on the poster. Then record it in the rectangle.



P 4

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

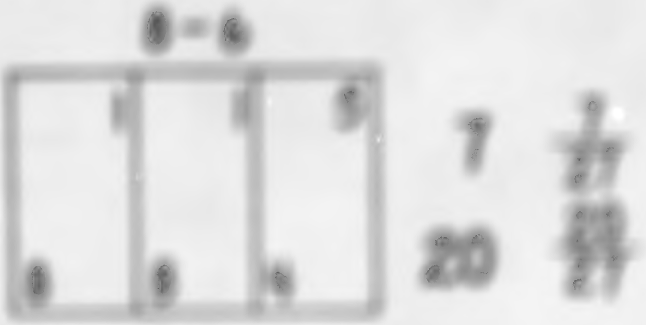
[illegible]

[illegible]

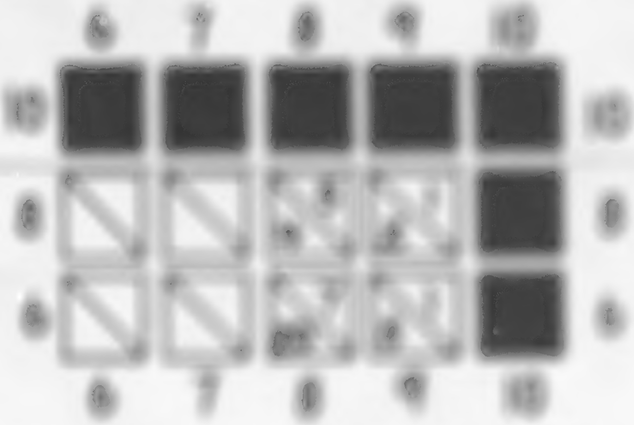
[illegible]

[illegible]

[illegible]


8 = 6
1
1
5
8
8
6
7
20
$\frac{7}{27}$
$\frac{20}{27}$

[illegible]


6
7
8
9
10
10
10
8
8
6
6
6
7
8
9
10

7. [illegible]

[illegible]

[illegible]

P 6

[illegible]

[illegible]

[illegible]

[illegible]

6
7
8
9
10
6
7
8
9
10

[illegible]

[illegible]

- [illegible]
- [illegible]
- [illegible]

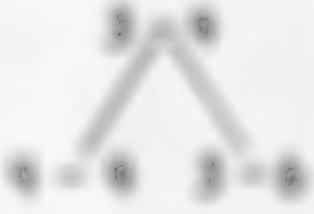
[illegible]

[illegible]

[illegible]

[illegible]

[illegible]



[illegible]

P 6

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Accept the explanations of several students about Rita's and Bruce's chances of winning starting from certain intermediate scores. If necessary, ask the following questions.

T (pointing to the square "$\boxed{\overset{7}{20}}$" for the score 8-6): What do the blue "7" and the red "20" in this square mean?

S: If the score is 8-6, Bruce has seven chances and Rita has twenty chances to win.

T: If the score is 9-6, what are Bruce's and Rita's chances to win?

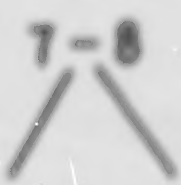
S: Bruce has one chance and Rita has eight chances to win.

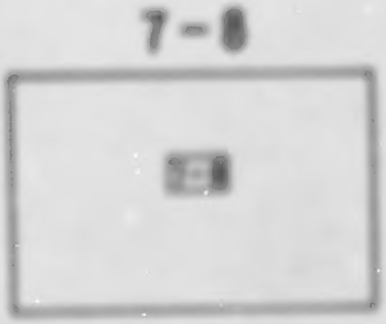
Distribute Worksheet P11. Instruct the students to copy the results on the poster to their worksheets.

NOTE: Four entries were made in the poster's chart during Lesson P3.1. Proceed to fill in the other four locations. The corresponding intermediate score is indicated before the description of an activity.

GAME IS AT 7-8

Draw this card picture on the board. Also draw a rectangle and place the score-card $\boxed{7-8}$ inside of it.





P 5

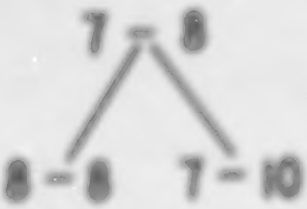
T: One day, Rita and Bruce stopped playing when the score was 7-8. Who is favored to win when they later continue this game?

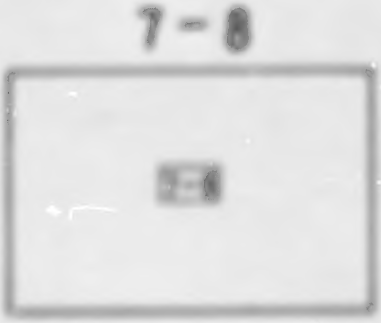
S: Bruce because he wins if the blue marble is chosen next, whereas Rita wins only if a red marble is chosen the next three times.

T: Let's calculate the winning chances of each player if they continue the game the next day. The score is 7-8; so what could the next score be?

S: 8-8 if a red marble is chosen, or 7-10 and a win for Bruce if the blue marble is selected.

Add this information to the card picture.





T: What are the chances for each of these scores?

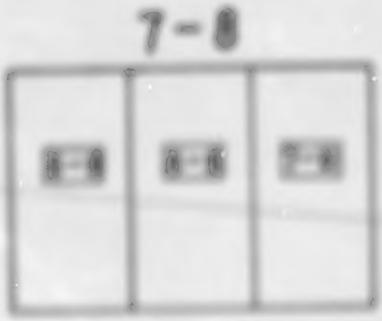
S: Two chances for 8-8 and one chance for 7-10 since there are two red marbles and one blue marble.

S: $\frac{2}{3}$ for 8-8 and $\frac{1}{3}$ for 7-10.

T: How can I show this on the rectangle?

S: Divide the rectangle into three pieces all the same size. Put $\boxed{8-8}$ in two parts and $\boxed{7-10}$ in one part.

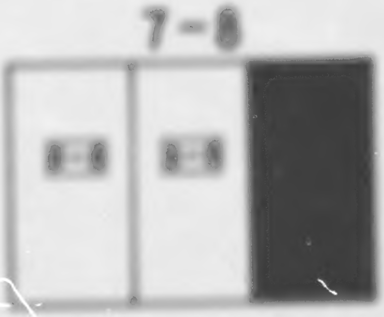
Invite a student to do the partitioning and suggest using a meter stick for accuracy. For example, if one side of the rectangle is 74 cm, the width of each of the three parts should be about 25 cm.



T (pointing to the region for 7-9): What happens if the score is 7-9?

S: Bruce is the winner.

T: Color the region with 7-9 in it blue.



T (pointing to the regions for 8-8): What do we know if the score is 8-8?

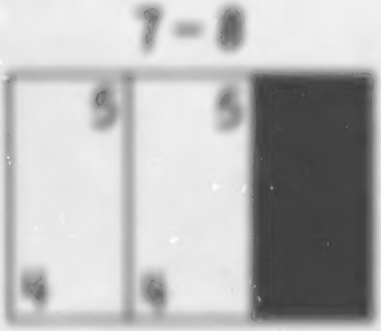
S: The next score could be 9-8 or 8-9. Divide each region for 8-8 into three parts.

T: Yes, I could do that but do I have to do all that work?

S: No, the results for 8-8 are already recorded on the poster. Bruce has five chances to win and Rita has four chances.

P5

Indicate where this result is recorded on the poster. Write this information on the board.



T: Into how many pieces should we divide each of these three regions?

S: Nine.

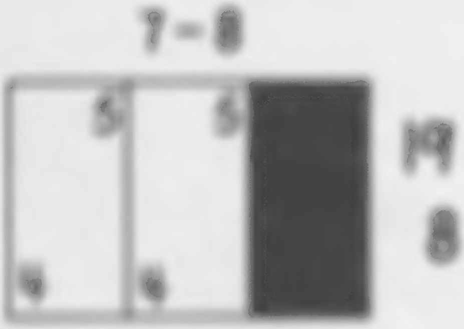
T: How many pieces would there be altogether?

S: Twenty-seven; 3 × 9 = 27.

T: How many would be colored red? . . . blue?

S: Eight would be colored red; 4 + 4 = 8. Nineteen would be colored blue; 5 + 5 + 9 = 19.

Write this information on the board.



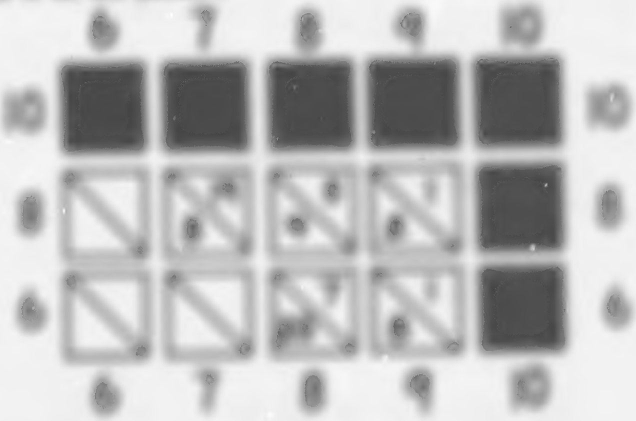
T: Therefore, if the score is 7-6, Bruce is more than twice as likely to win as Rita is because 19 is more than 2 x 8 (16). What is Bruce's probability of winning when the score is 7-6?

S: $\frac{19}{27}$ or 19 out of 27.

T: What is Rita's probability of winning?

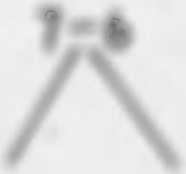
S: $\frac{8}{27}$ or 8 out of 27.

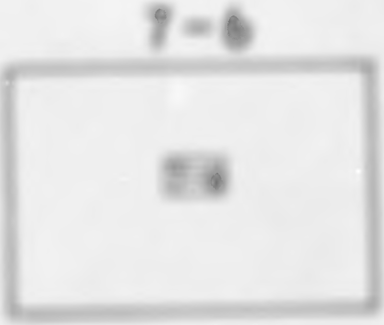
Ask the students to record the result for 7-6 on their worksheets and invite a student to write it on the poster.



score is at 7-6

Draw this card picture on the board. Also draw a rectangle and put the score card [7-6] inside of it.

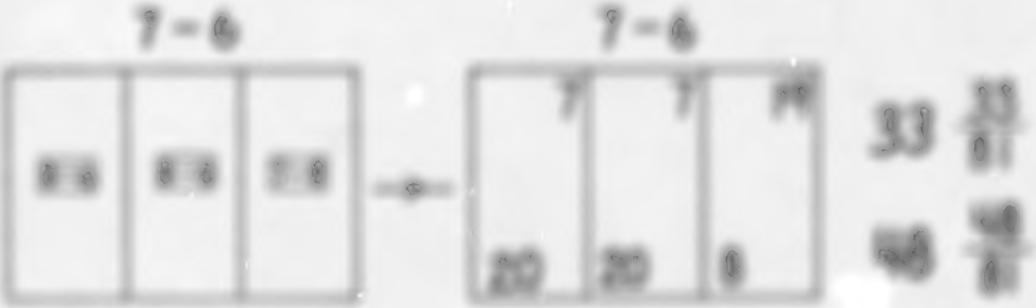




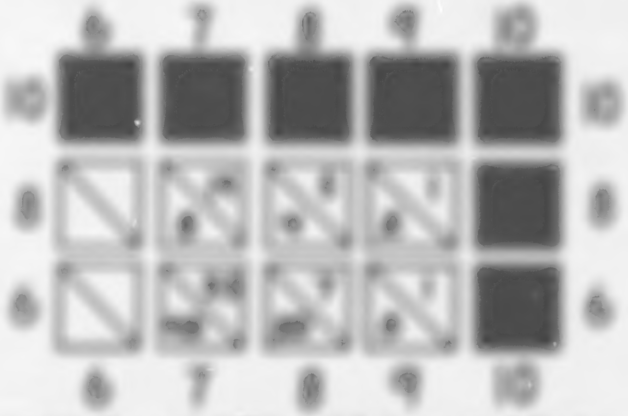
P5

T: Copy this cord picture and draw a rectangle. Then calculate [illegible] and Bruno's chances to win when the score is 7-6.

Let the students work independently for a while and assist any student who needs help. Encourage students to use the results already recorded on the poster. Following student suggestions, solve the problem collectively at the board. The next illustration indicates the steps your class might suggest.



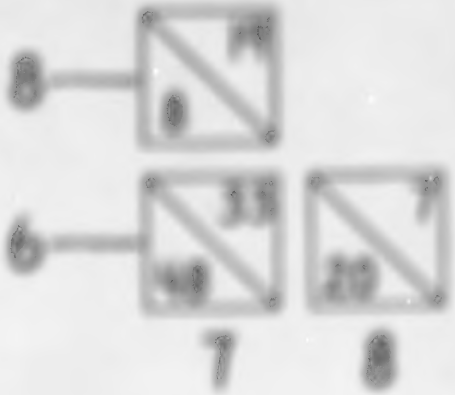
Ask the students to record the result for 7-6 on their worksheets and invite a student to write it on the poster.



T: To calculate the probabilities of 7-6, we used the known results for two other scores. Which two scores?

S: 8-6 and 7-7.

[illegible]



[illegible]

[illegible]

[illegible]

a. [illegible]

b. [illegible]

8. Each red number in the top row is a divisor of the red number immediately below it; that is, 2 is a divisor of [illegible], 3 is a divisor of [illegible], and 4 is a divisor of [illegible]. In fact [illegible]. We multiply by a number 1 larger each time.

9. To find a red number in the second row, double the red number to the right of it and add the red number above it. For example, [illegible]. The same relationship is true for the blue numbers: [illegible].

[illegible]

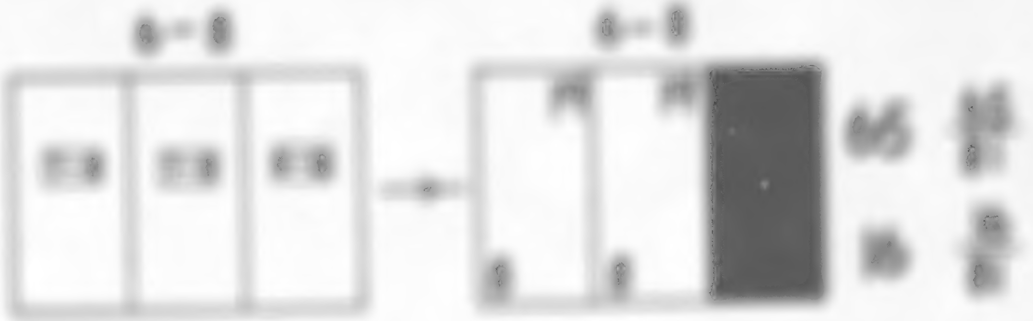
T (pointing to the square for [illegible]): Some of these patterns can be useful in predicting the winning chances when the score is [illegible]. How many winning chances do you predict for Max when the score is [illegible]? . . . for Bruce?

On the board, list the predictions for Max and for Bruce in red and blue, respectively. Encourage students to use any patterns that were just described. For example, a student might predict that Max's chances are [illegible] due to the pattern: 2, 3, 4 among the red numbers in the top row. And a student might then predict that Bruce's chances are [illegible] and the sum of the two numbers in that square should be [illegible].

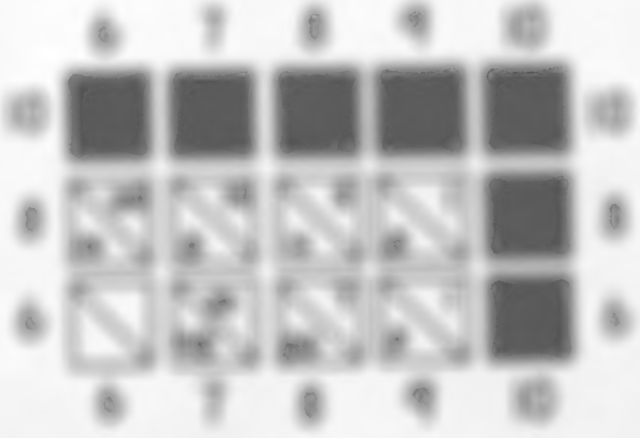
Draw this picture. Then draw a rectangle and place the scorecard [illegible] inside of it.

[illegible] Let's check if some of your predictions are correct. Copy this [illegible] [illegible] and use it to calculate [illegible] [illegible] [illegible] to [illegible] the [illegible] in 8-8. Remember to use known results if you can.

After your students have calculated the missing [illegible] for 8-8, use student suggestions to solve the problem collectively at the board. The next illustration indicates the steps your class might suggest.



Comment on whether any of the predictions were correct. Invite a student to record the result for 8-8 on the poster.



P 5

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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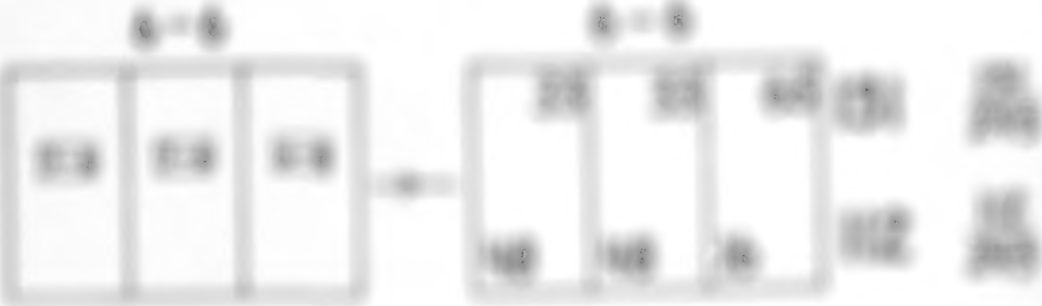
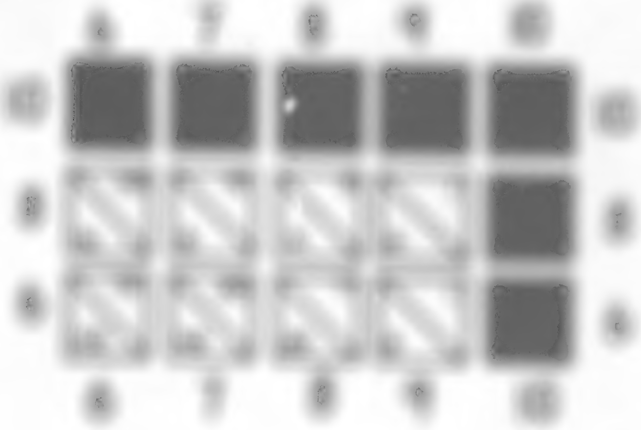
[illegible]



[illegible]

[illegible]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[illegible]

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